CS 560: Homework 2: Foundations
Polyhderal Domains, Dependences & Operations

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The goal of this assignment is to gain experience in the mathematical objects—affine functions, and polyhderal domains—that we manipulate, and some of the operations we perform on them. You will also gain familiarity with the multiple alternate representations of these objects.

Problem I: (non parametric and parametric) affine functions: [40 pts]

You are given the following defintions:

- $f_0(i,j) = j, i-1$, written in the "dependence notation" as $f_0 = (i,j \rightarrow j, i-1)$
- $f_1(i,j,k) = i-j+2k, j+k, k+2$, or $(i,j,k \rightarrow i-j+2k, j+k, k+2)$,
- $f_2(i,j,k) = k, j, k$, or $(i,j,k \rightarrow k, j, k)$,
- $f_3(i,j) = i+j, j-2$, or $(i,j \rightarrow i+j, j-2)$,
- $f_4(i,j) = i+j, N-i+1$, or and $(i,j \rightarrow i+j, N-i+1)$, which is parameterized with a single parameter, N.

1. Write the arity of each of the functions. For example, $f_0 : \mathcal{Z}^2 \rightarrow \mathcal{Z}^2$
2. Write each of the functions in matrix notation. For example,

$$f_0(z) = f_0\begin{pmatrix} i \\ j \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} i \\ j \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

3. Write down the (relational) inverse of each function.[1] For example, $f_0^{-1}(i,j) = j+1, i$
4. Find each of the following functions (note that all the functions may not be defined). Write your answer in the "dependence notation" as well as the matrix notation. Remember that `@` denotes left-associative function composition, while $\circ$ and . denote the more mathematically common (right-associative) function composition.
   (a) $f_1 @ f_2$
   (b) $f_2 . f_1$
   (c) $f_2 @ f_1$
   (d) $f_1 @ f_2 @ f_3 @ f_4$
   (e) $f_1 \circ f_2 \circ f_3 \circ f_4$
   (f) $f_4 @ f_3$

[1] When a linear/affine function ($\mathcal{Z}^n \rightarrow \mathcal{Z}^m$ for $n \geq m$) does not admit a standard inverse, the function is many-to-one, and its range is a *subspace* of $\mathcal{Z}^m$. For any point in the range of the function, there is a *set* of points in the domain. This is called the *relational inverse*. For example, the inverse of $(i,j \rightarrow i-1)$ is given by $f^{-1}(p) = \{(i,j) \in \mathcal{Z}^2 \mid i = p+1\}$. Note that there are no constraints on j, and this is a straight line in $\mathcal{Z}^2$.

Problem II: (non parametric and parametric) domains & polyhedra [50 pts] You are given the following domains:

- $D_1 = \{i, j \mid 0 \leq i - j < 100 \ \&\& \ 0 \leq i < 100\}$
- $D_2 = \{i, j, k \mid 0 \leq (i, j) < N \ \&\& \ 0 \leq k \leq (i, j)\}$. It has a single parameter, N.
- D_3 is the polyhedron defined by the vertices $[0,0]$, $[-9,9]$, $[0,18]$, $[50,0]$, $[59,9]$, and $[50,18]$.
- D_4 is the polyhedron, parameterized by N, defined by the vertices $[0,0,N]$, $[0,N,0]$, $[N,0,0]$, $[0,0,-N]$, $[0,-N,0]$, and $[-N,0,0]$.
- D_5 is the polyhedron, parameterized by N, defined by the constraints $\{i, j \mid 0 \leq (i, j) \&\& i + 2j < N\}$.

Recall that there are two representations of polyhedra, the constraint form, and the generator (vertex/ray) form. Of these, the constraint form can be written either as a set, or in the matrix representation. So we have three ways to describe polyhedra. Also let $f_4'(i, j) = i + j, 100 - i$.

1. Write down the two missing forms of the above polyhedra. Note, D_5 is tricky.
2. Compute `PreImage(`D_1, f_3`)` (and report your answer using all three notations).
3. What is `Image(`D_1, f_4'`)` (do not use the vertex representation of D_1, but rather, use the trick discussed in the class notes).
4. What is `Intersection(PreImage(`D_1, f_4'`), Image(`D_1, f_3`))`
5. Determine `Intersection(PreImage(`D_2, f_2`), Image(`D_2, f_1`))`

Report [10 pts] Write a short summary of what you learnt in doing this assignment. Submit everything as a single pdf file called `HW2.pdf` via Checkin.