Simplifying Reductions

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Reductions

- A reduction is an associative and commutative operator applied to collections of values to produce a collection of results

Reductions

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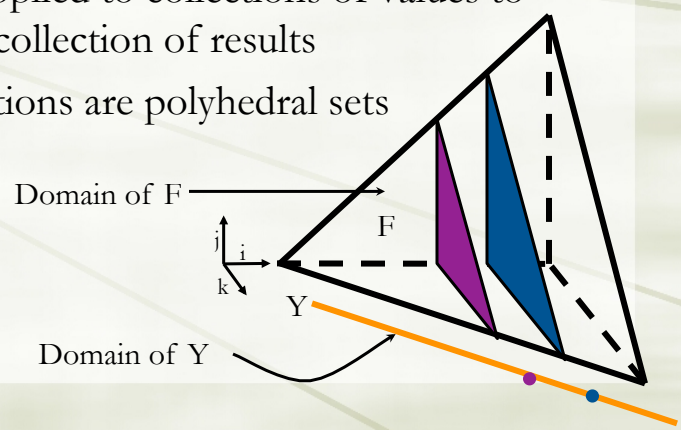
- Our collections are polyhedral sets



Highlights

- Simplify Reductions
  - Automatically
  - Optimally
- Non-trivial Examples
  - RNA Secondary Structure Prediction: (`mfold` package)
  - Finite Element Methods
  - Max filter
  - Line-of-sight computation in GIS

Outline

- Introduction and Problem Definition
- Sharing
- Simplification
  - Recursive Simplification
- Reduction Decomposition
- Motivating Example
- Distributivity
- Applications
- Previous Work and Conclusions

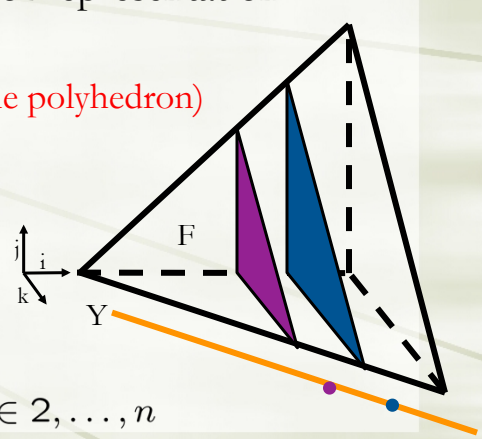
Representation

- Three equivalent forms of representation
  - Geometric
  - Loops (bounds define the polyhedron)

```
for i = 1 to n {
 Y[i] = 0;
 for j = 1 to i-1
  for k = 1 to i-j
   Y[i] += F[i,j,k]; }
```

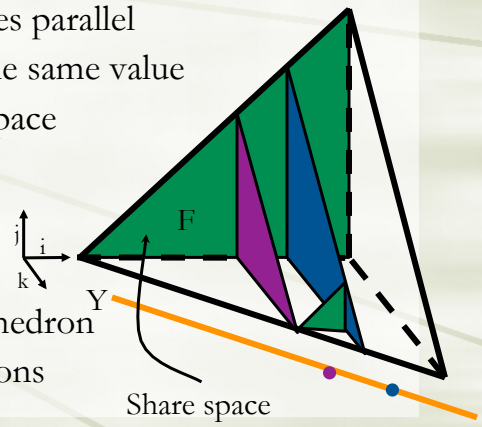
  - Equations

$$Y_i = \sum_{j=1}^{i-1} \sum_{k=1}^{i-j} F_{i,j,k}, \; i \in 2, \ldots, n$$

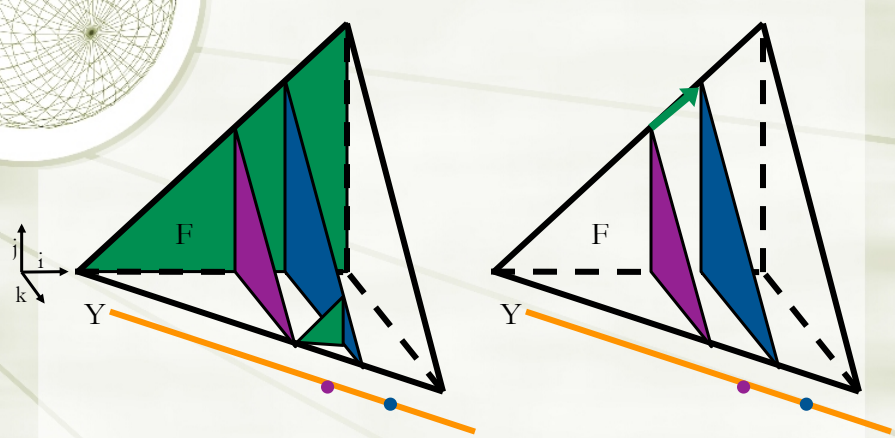


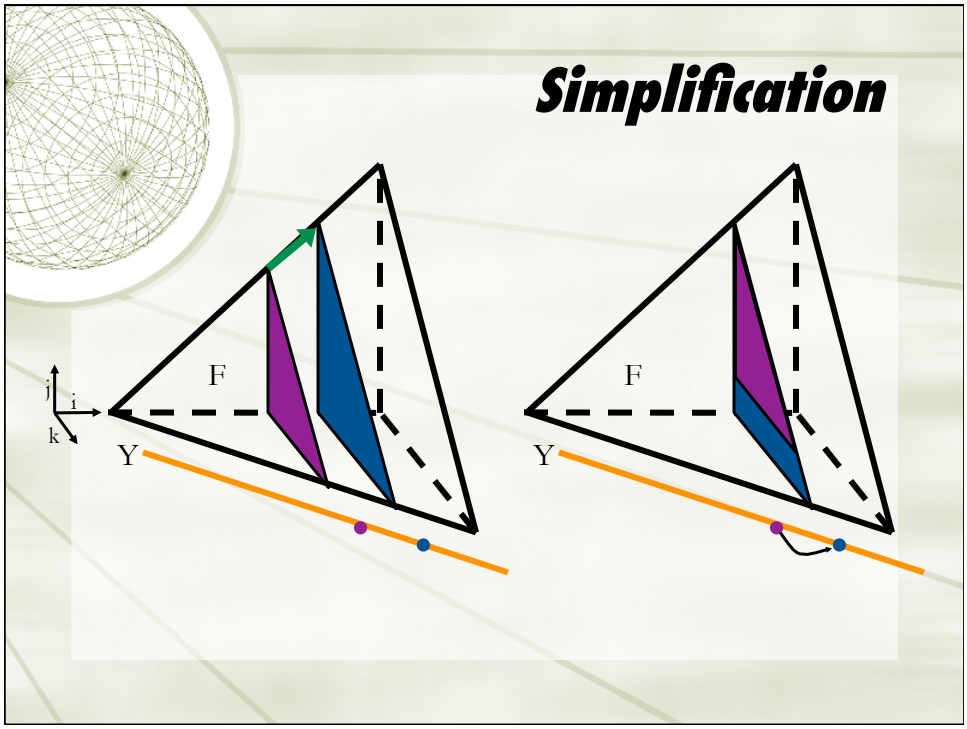
Sharing

- If $F_{i,j,k} = X_k$
  - All index points on planes parallel to the {i,j} plane have the same value
  - {i,j} is called the share space
    - Denoted by green
  - Aim to replace this polyhedron by one of lesser dimensions



Simplification

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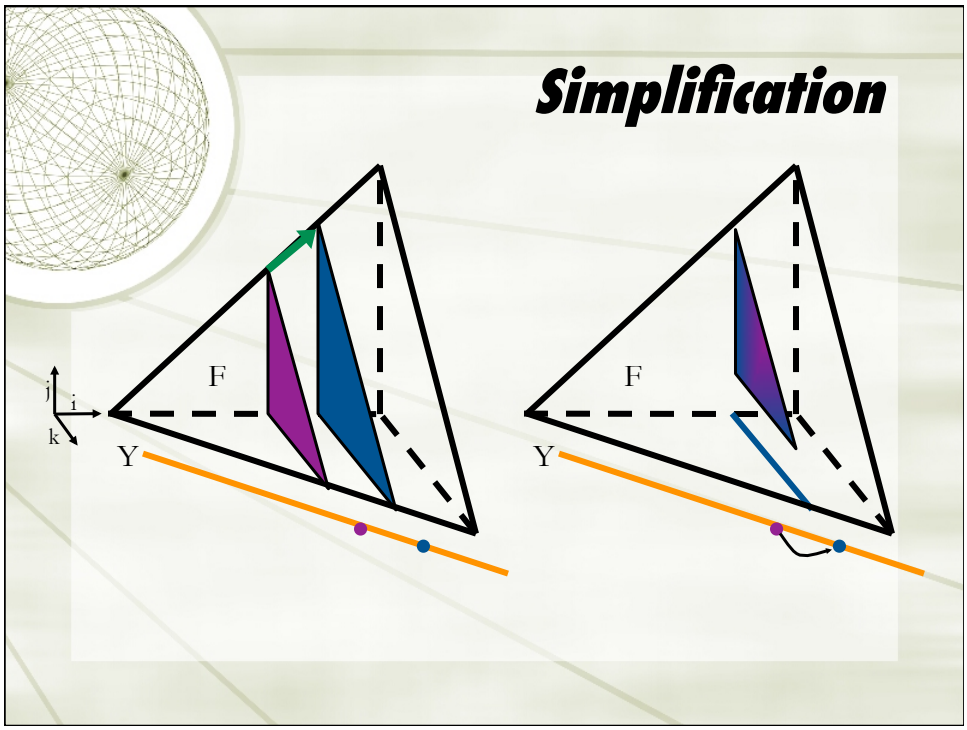
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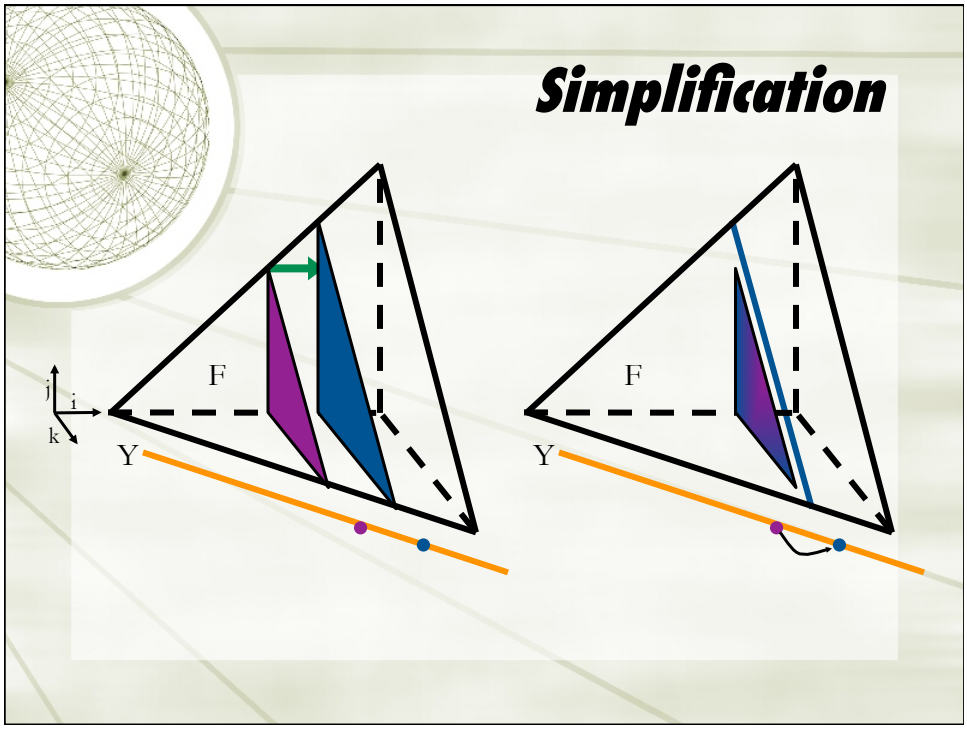
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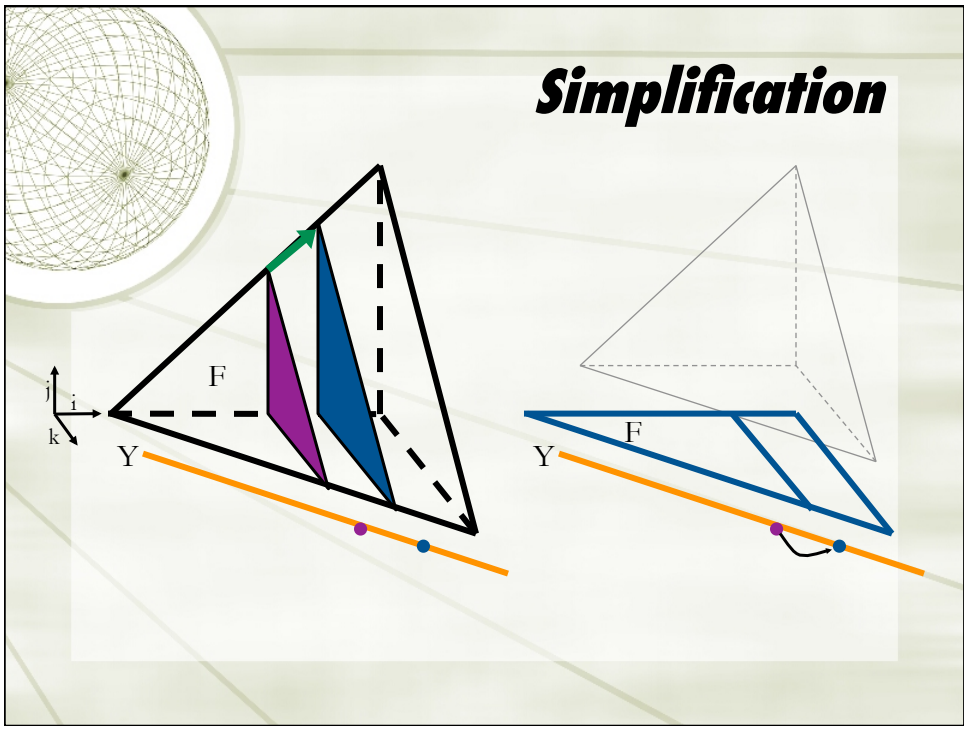
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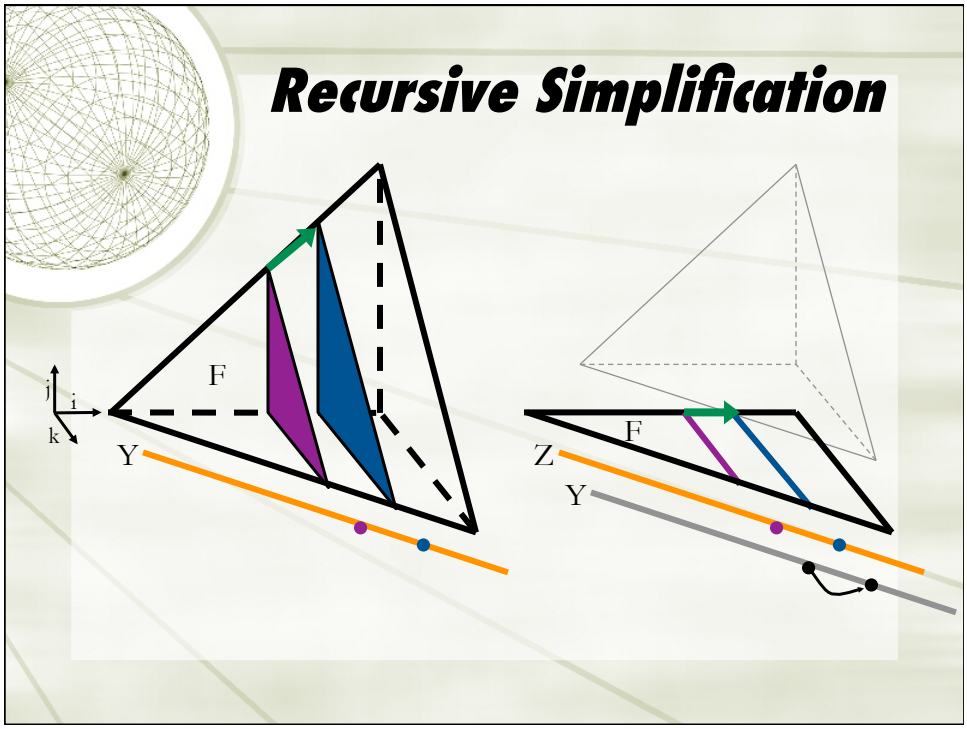
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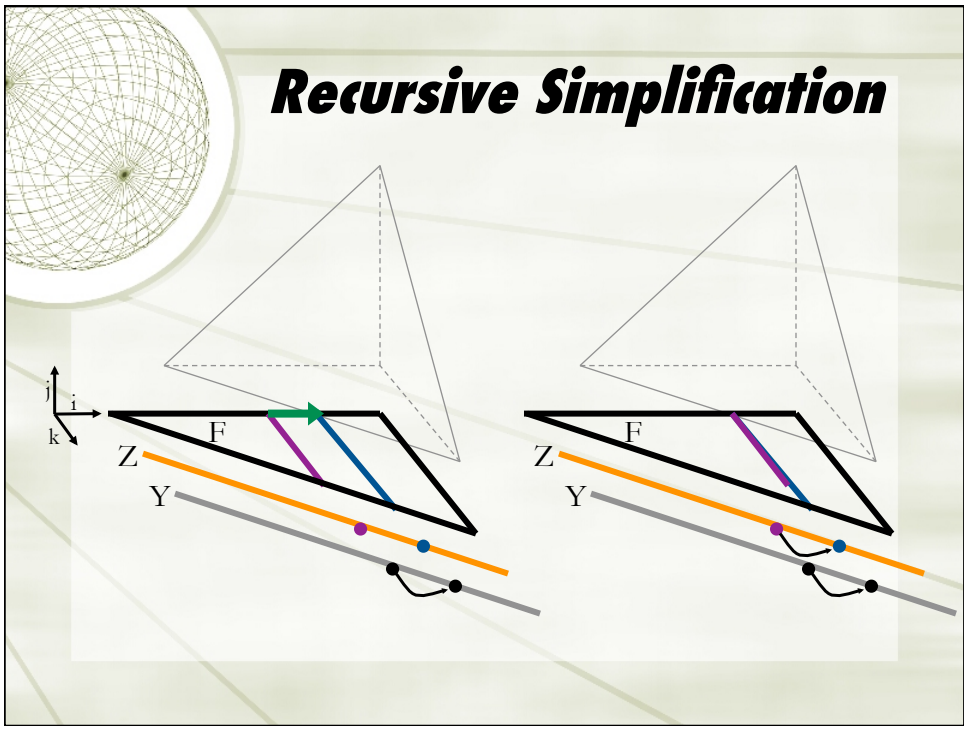
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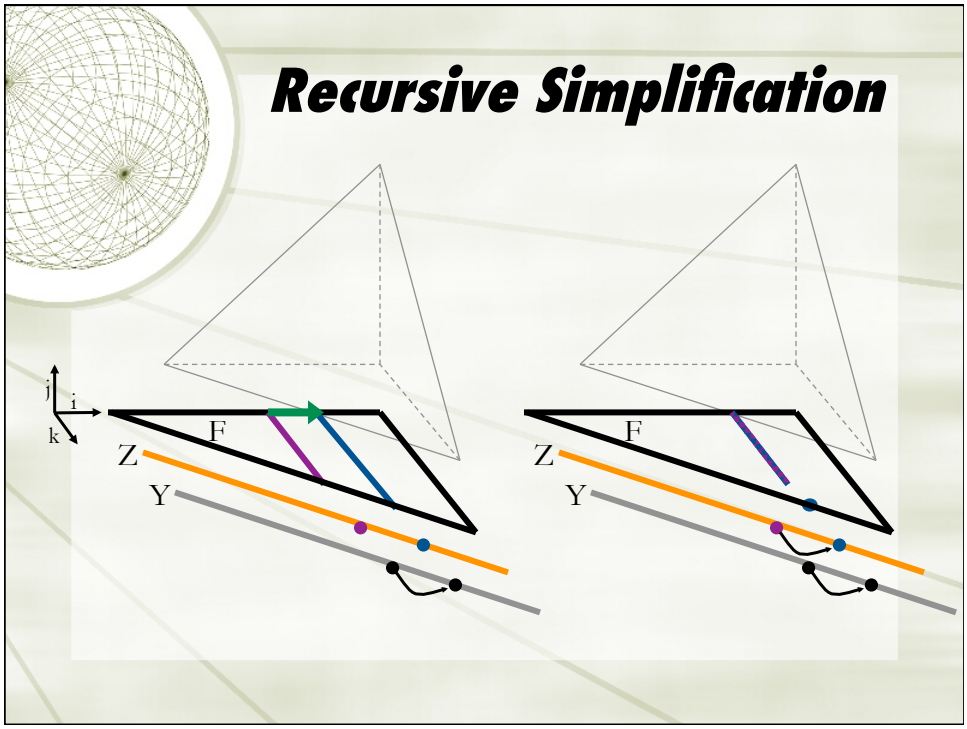
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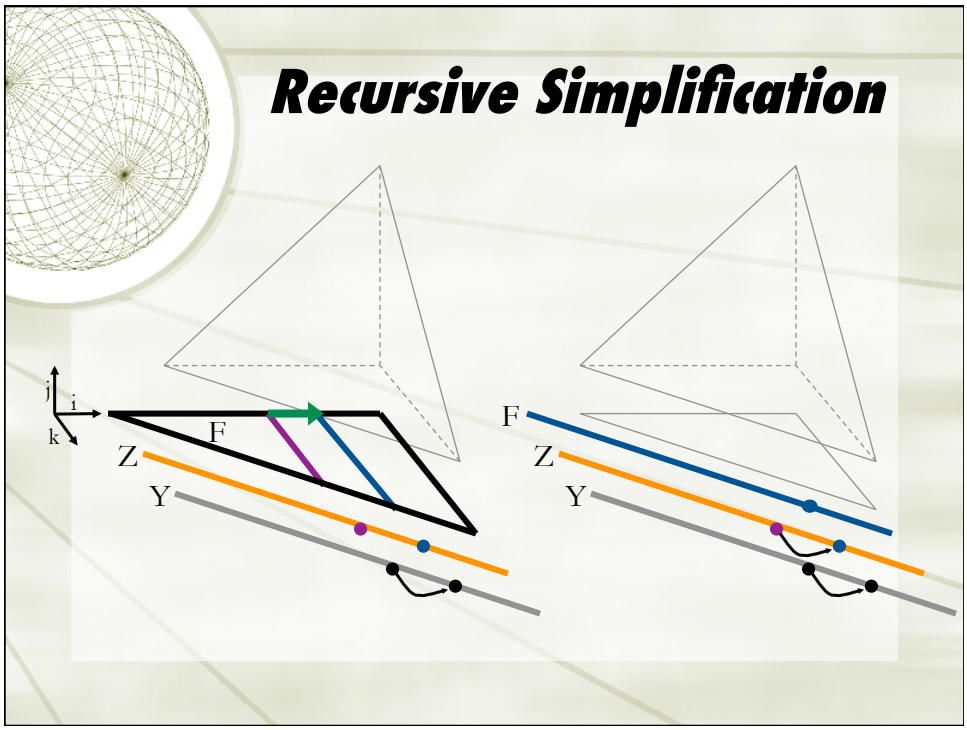
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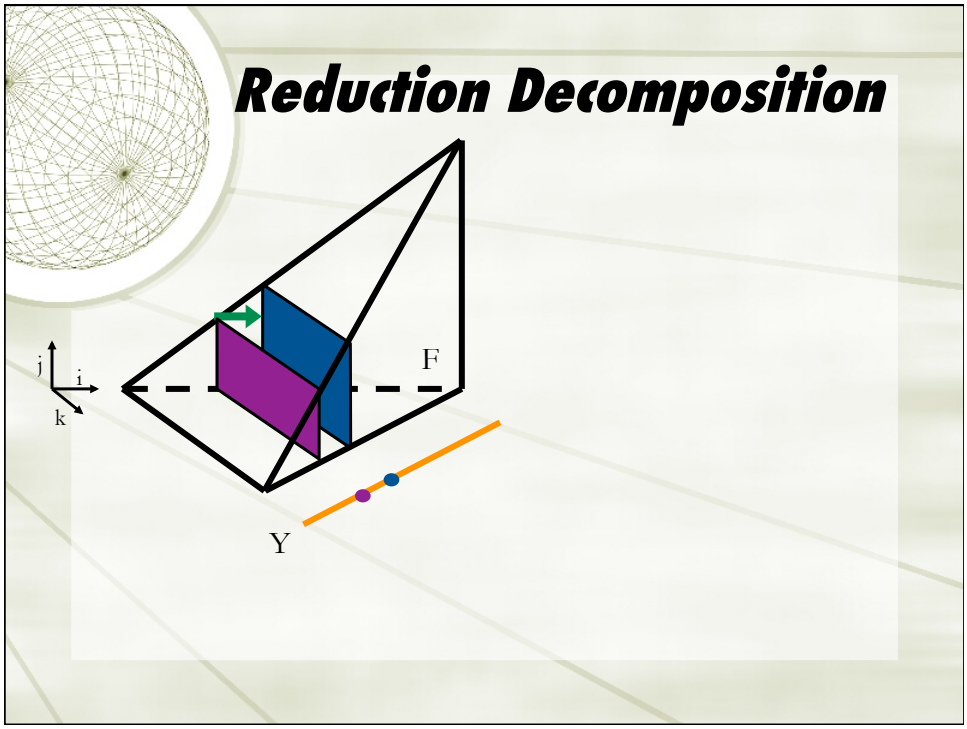
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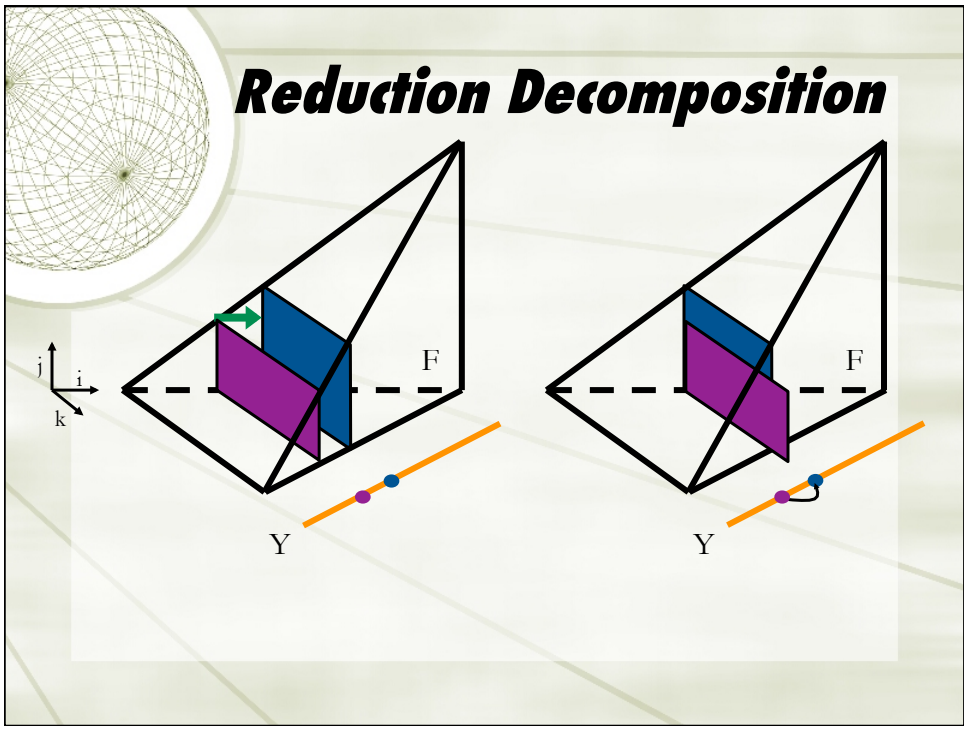
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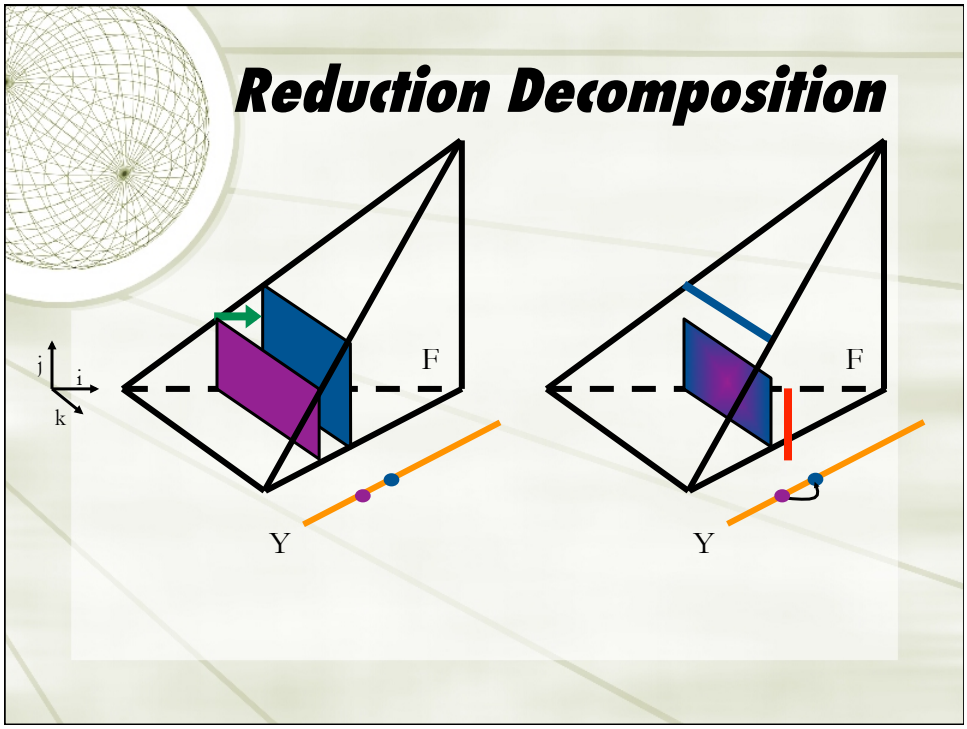
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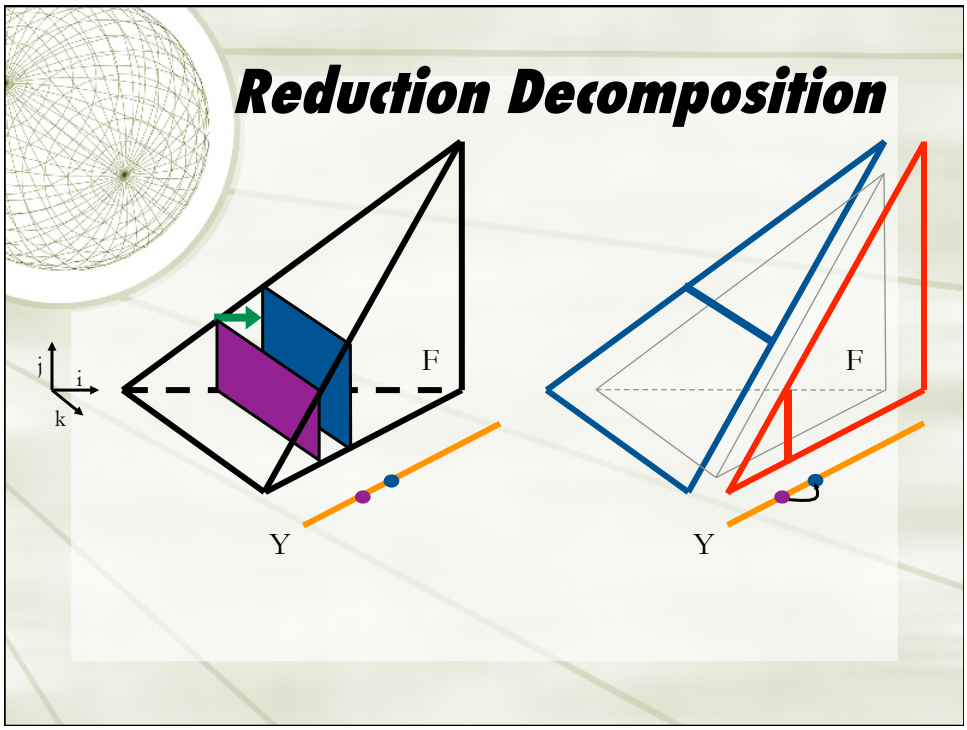
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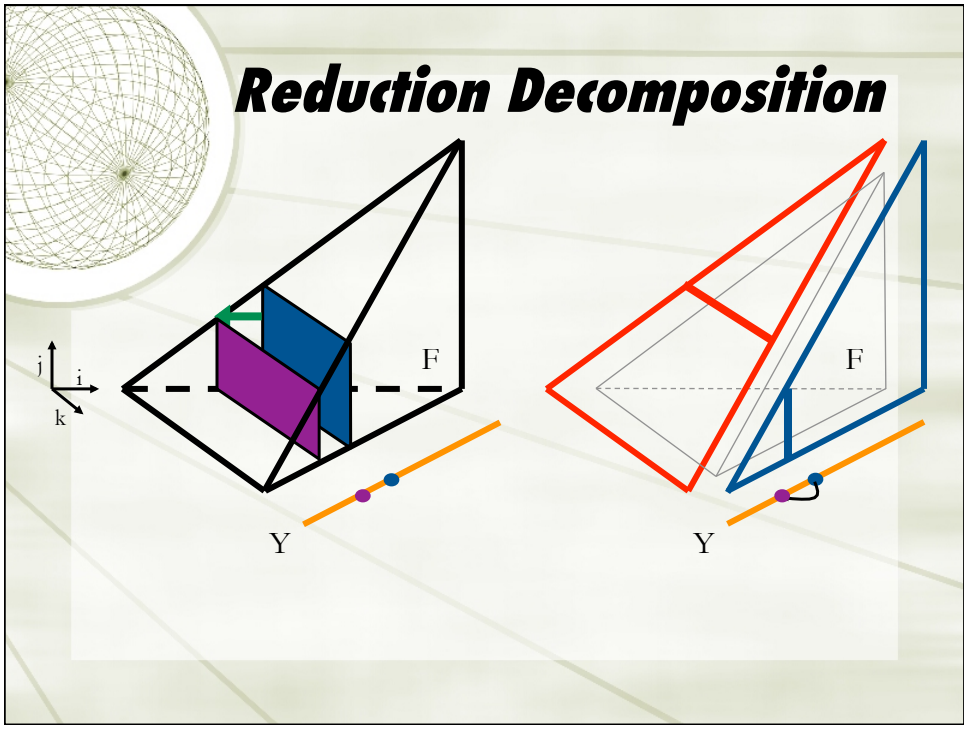
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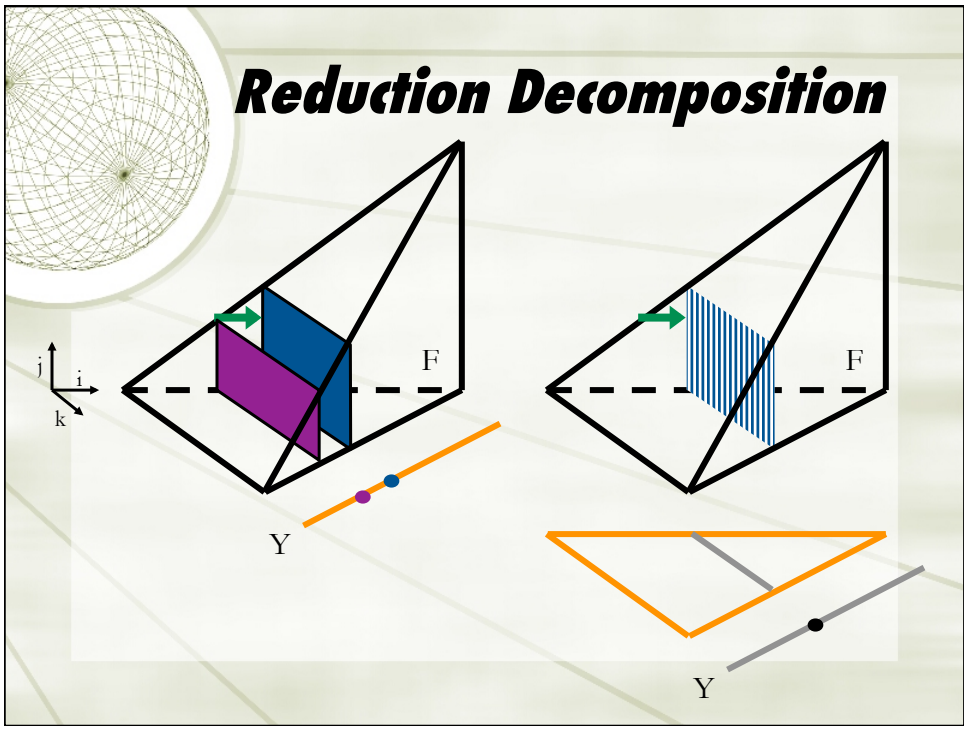
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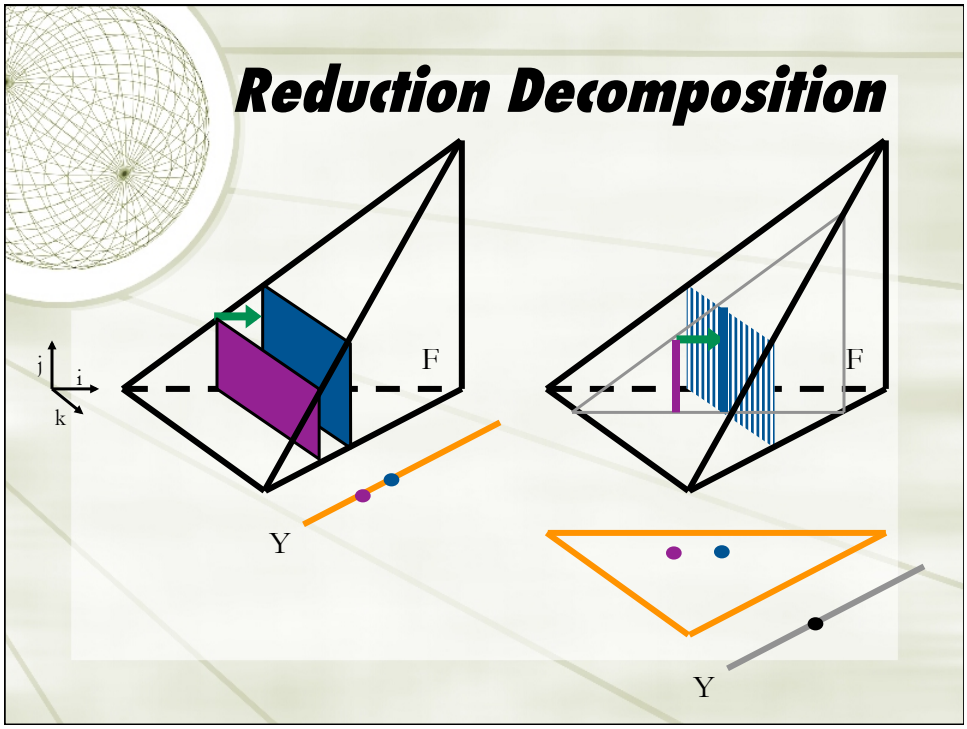
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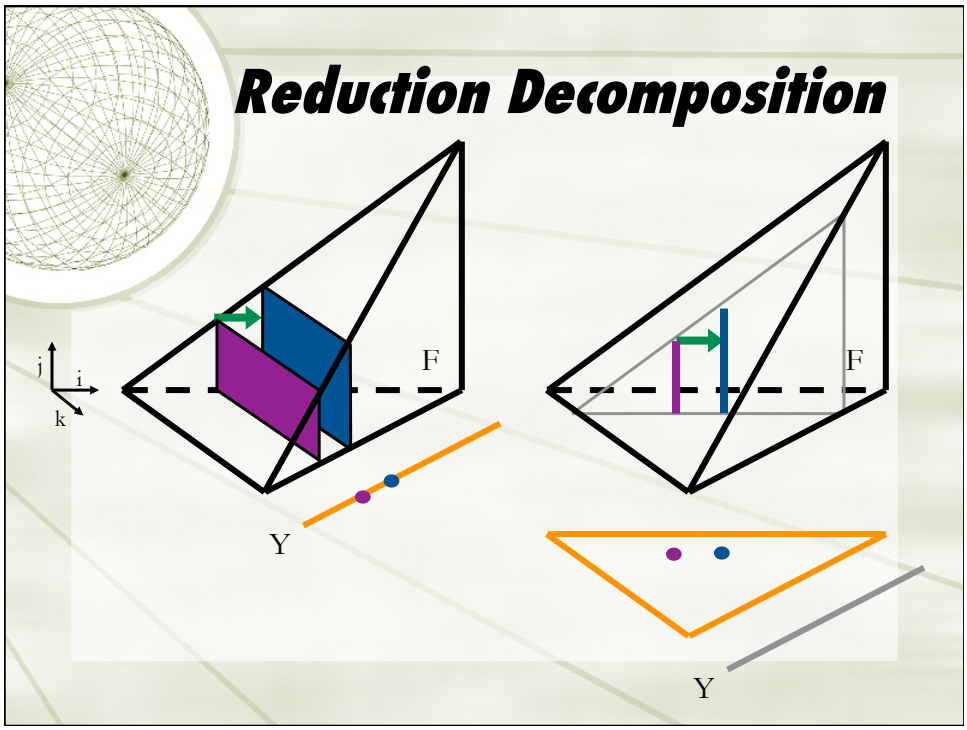
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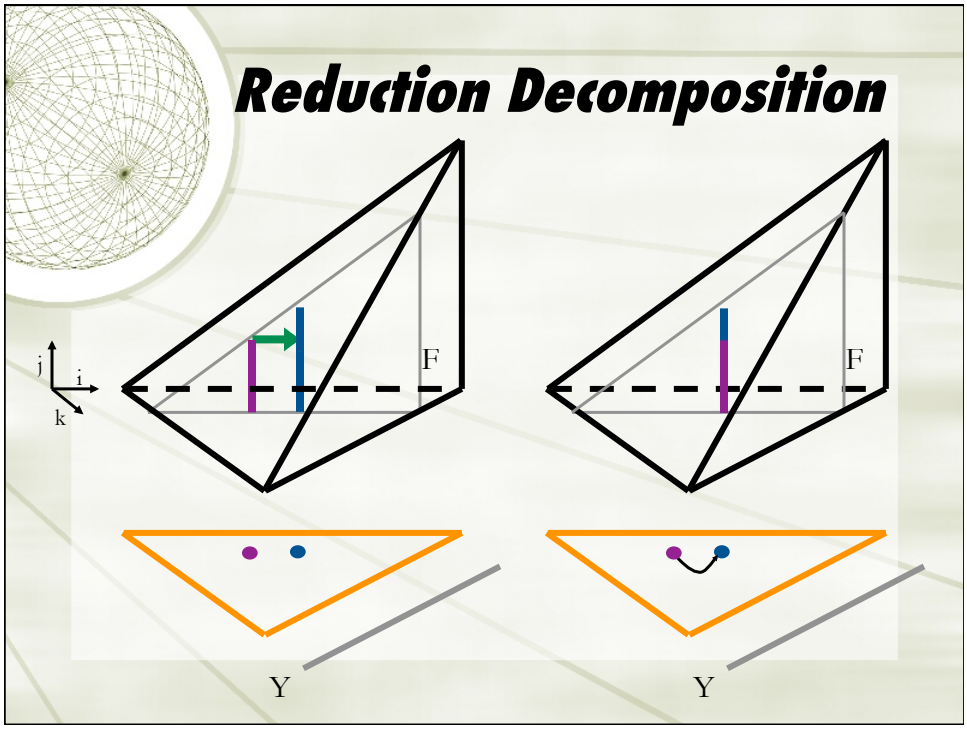
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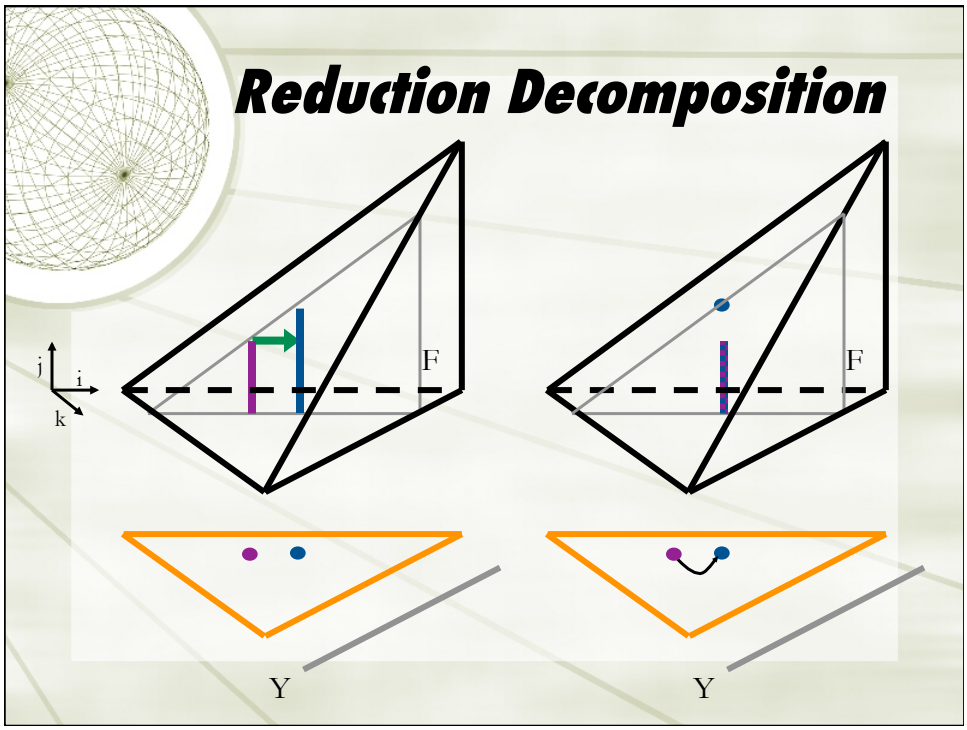
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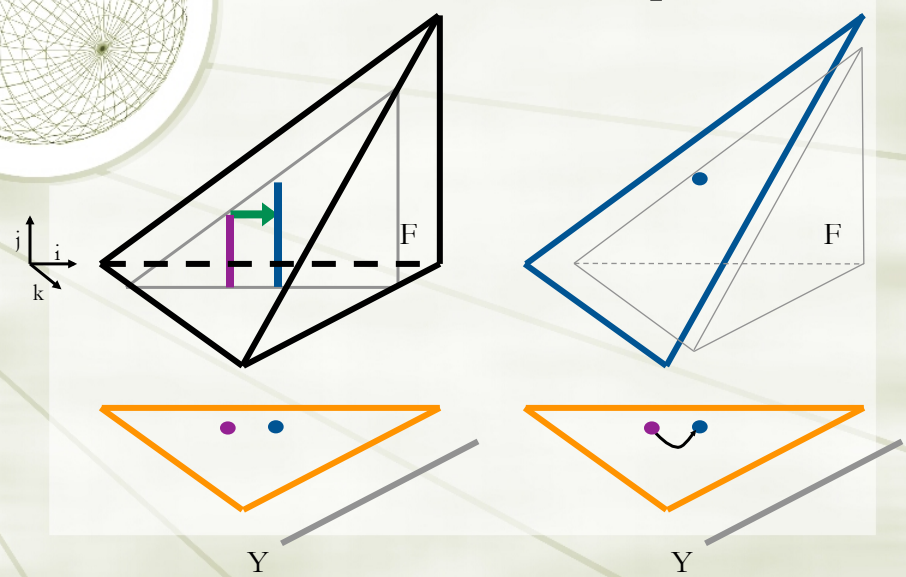

Motivating Example

- Equation

$$Y_i = \max_{j=i}^{2i} \max_{k=i}^{3i-j} X_{j,k}$$

- Loop

```
for i = 0 to n {
 Y[i] = -infinity;
 for j = i to 2i
  for k = i to 3i-j
   Y[i] = max(Y[i],X[j,k]); }
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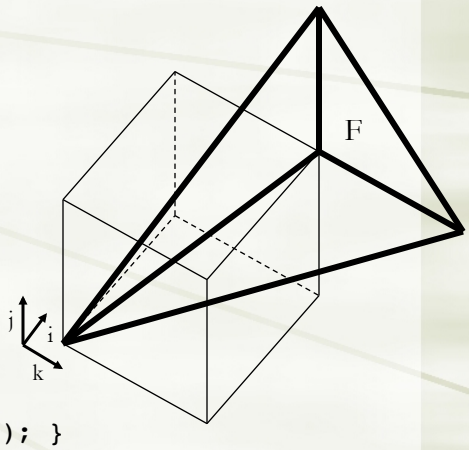

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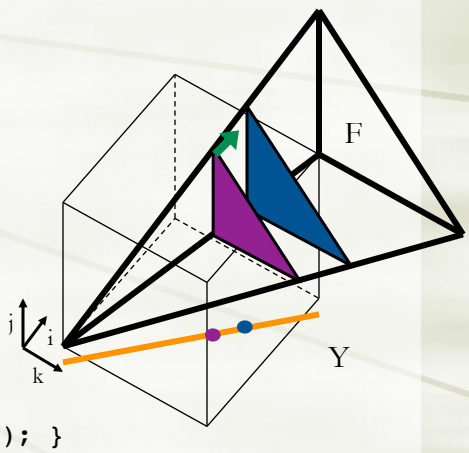

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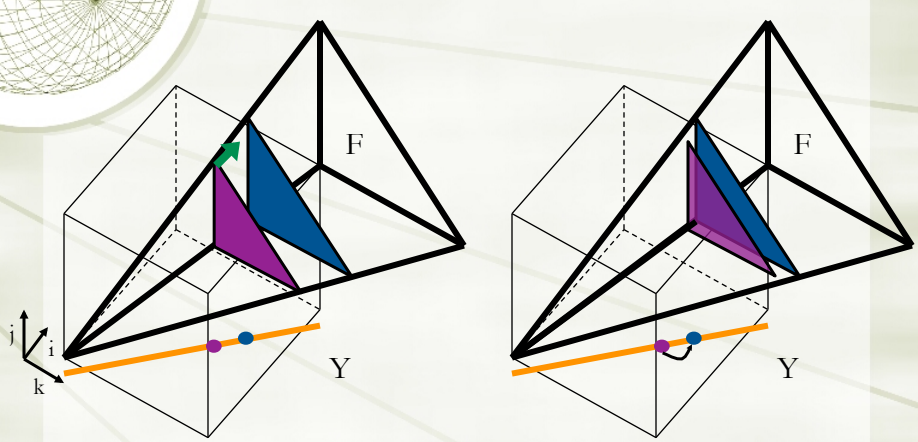

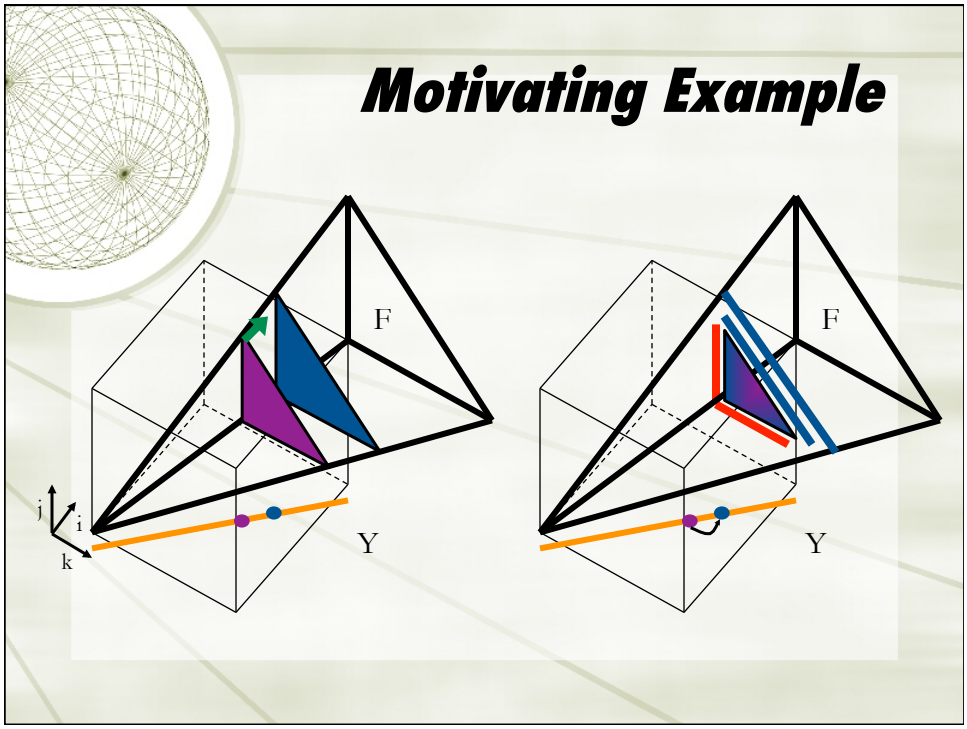
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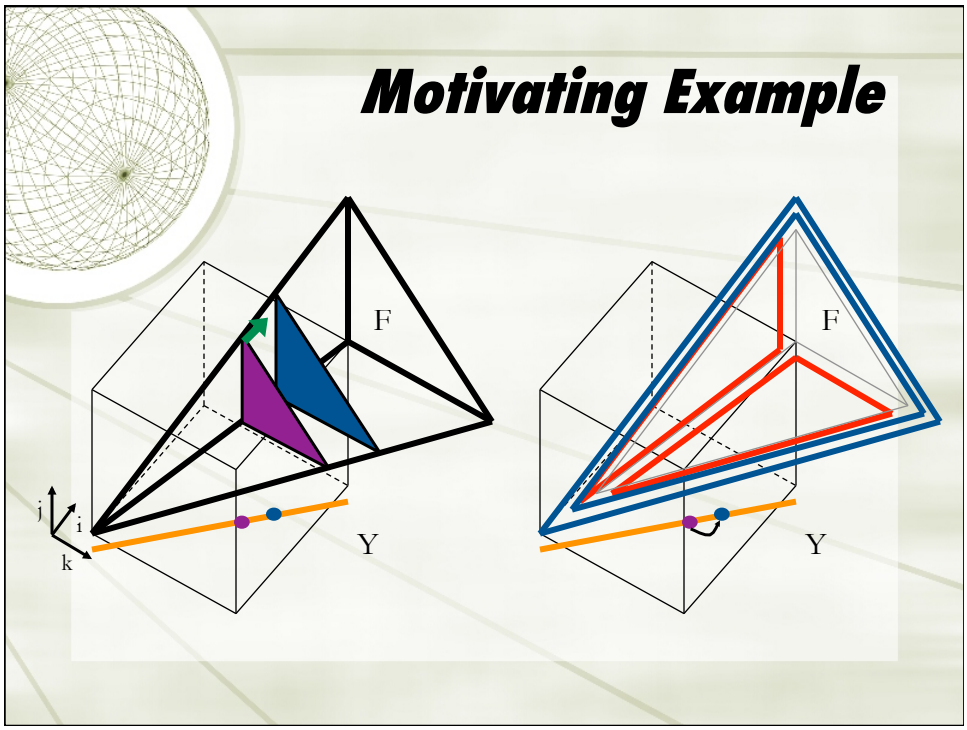
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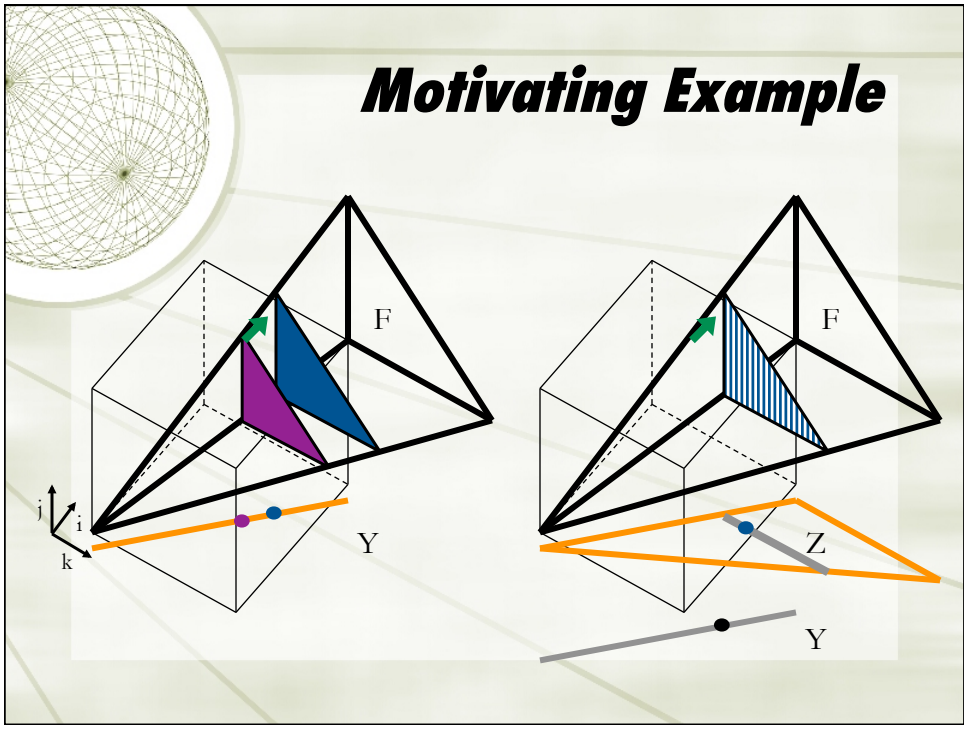
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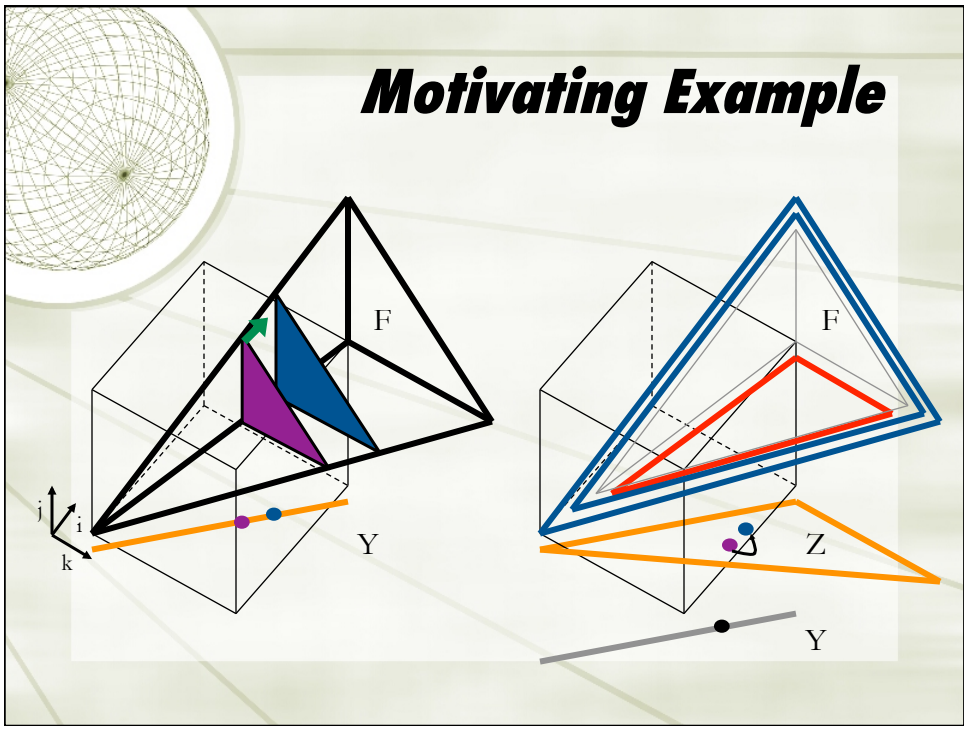
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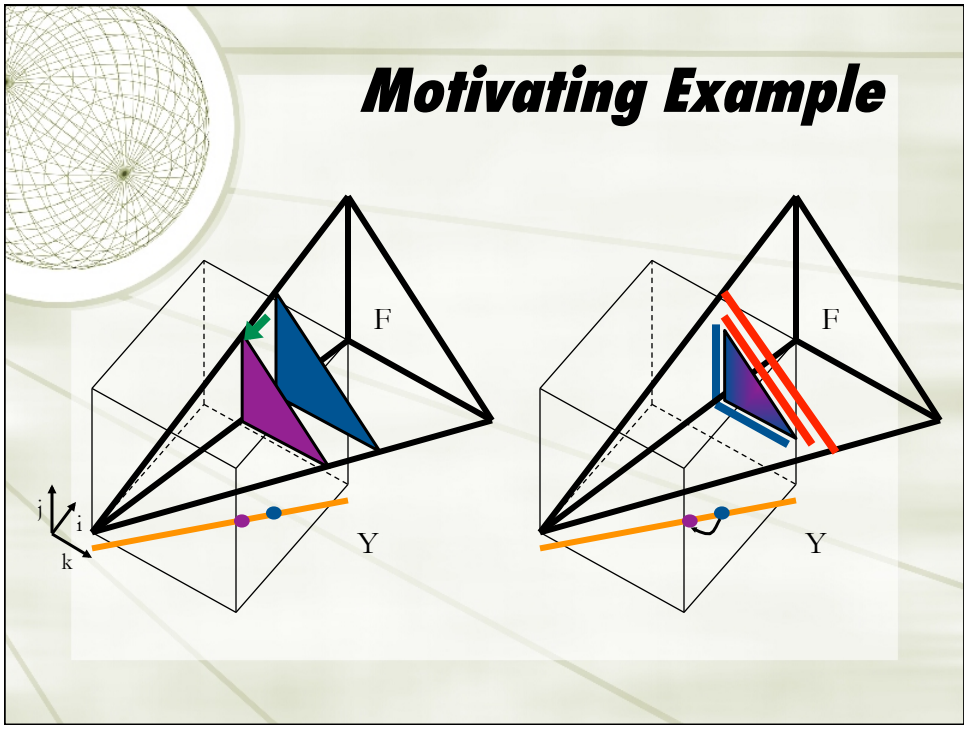
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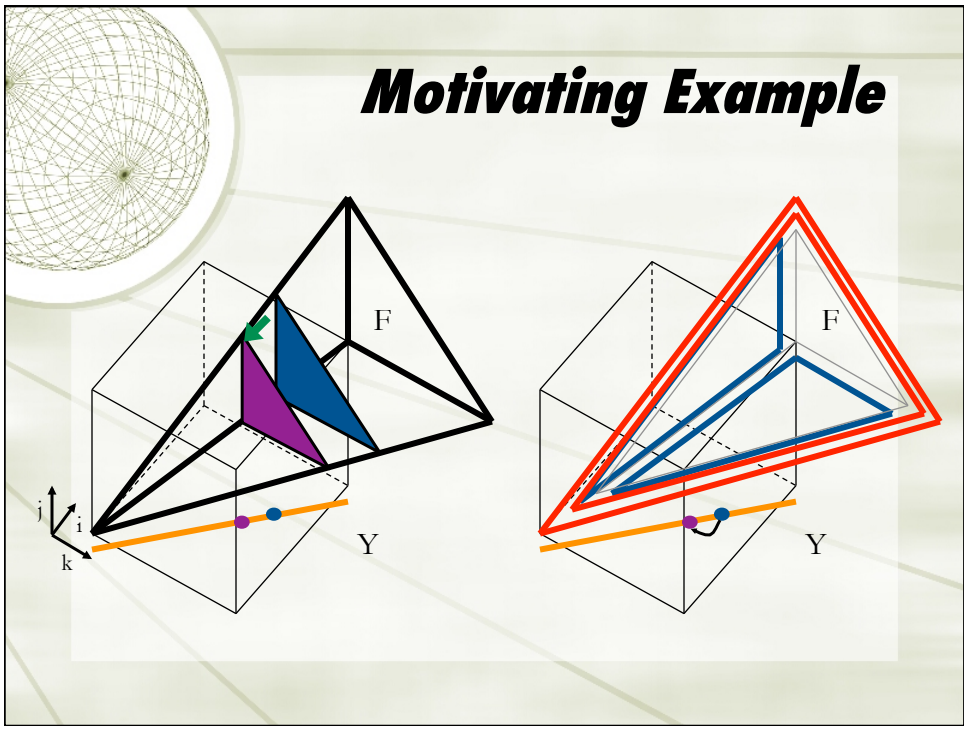
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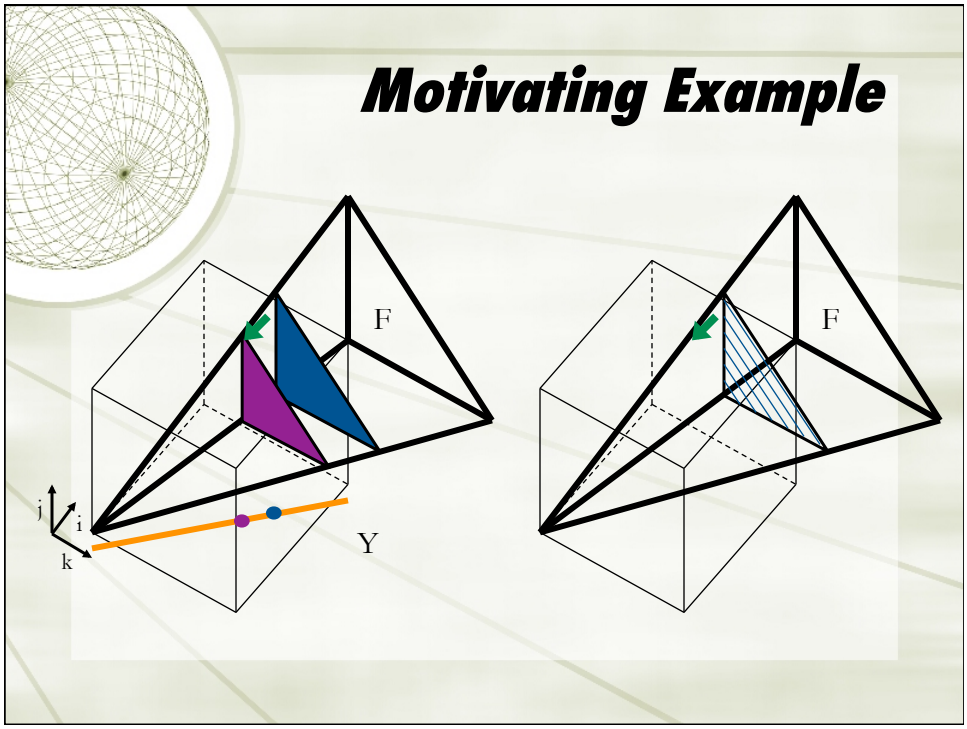
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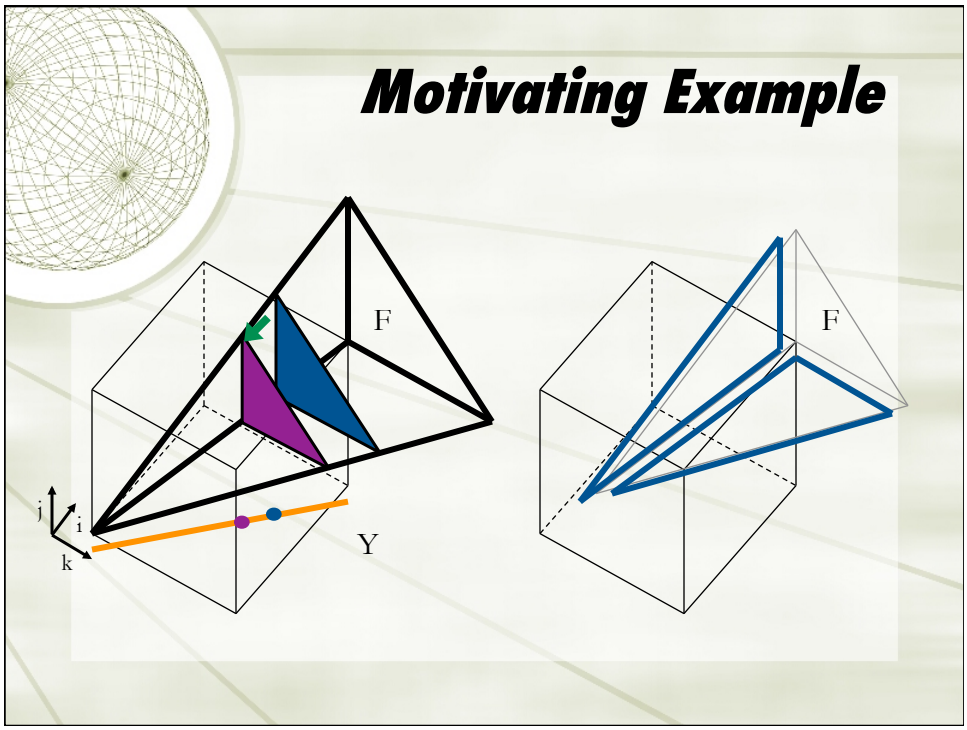
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Motivating Ex. Equations

- Let us now see the same transformations applied to equations and the resultant code

$$Y_i = \max_{j=i}^{2i} \max_{k=i}^{3i-j} X_{j,k}$$

- We saw that the inner reduction should be along the $j + k = \text{constant}$ direction
- Lets perform a change of indices to ensure this
  - Introduce $m = j + k$

$$Y_i = \max_{m=2i}^{3i} \max_{k=i}^{m-i} X_{m-k,k}$$

Motivating Ex. Equations (2)

$$Y_i = \max_{m=2i}^{3i} \max_{k=i}^{m-i} X_{m-k,k}$$

- Introduce temporary variable $Z_{i,m}$ to hold the result of the inner summation $\max_{k=i}^{m-i} X_{m-k,k}$

$$Y_i = \max_{m=2i}^{3i} Z_{i,m} \qquad Z_{i,m} = \max_{k=i}^{m-i} X_{m-k,k}$$

- When $m \geq 2(i+1)$, we get the following

$$Z_{i,m} = \max(\max_{k=i+1}^{m-(i+1)} X_{m-k,k}\ ,\ X_{m-i,i}\ ,\ X_{i,m-i})$$

Motivating Ex. Equations (3)

$$Z_{i,m} = \max(\max_{k=i+1}^{m-(i+1)} X_{m-k,k}\ ,\ X_{m-i,i}\ ,\ X_{i,m-i})$$

- Note that the max over k is simply $Z_{i+1,m}$
- We can replace the equation for $Z_{i,m}, m \geq 2(i+1)$ by $Z_{i,m} = \max(Z_{i+1,m}\ ,\ X_{m-i,i}\ ,\ X_{i,m-i})$
- The final set of equations

$$Y_i = \max_{m=2i}^{3i} Z_{i,m}$$

$$Z_{i,m} = \begin{cases} 2i \leq m < 2(i+1) & \max_{k=i}^{m-i} X_{m-k,k} \\ 2(i+1) \leq m \leq 3i & \max(Z_{i+1,m}\ ,\ X_{m-i,i}\ ,\ X_{i,m-i}) \end{cases}$$

Motivating Ex. Equations (4)

- The code for the transformed set of equations

```
for i = n downto 0 {
 for m = 2i to 2i+1 {
  Z[i,m] = -infinity;
  for k = i to m-i
   Z[i,m] = max(Z[i,m],X[m-k,k]); }
 for m = 2i+2 to 3i
  Z[i,m] = max(Z[i+1,m],X[m-i,i],X[i,m-i]);
 Y[i] = -infinity;
 for m = 2i to 3i
  Y[i] = max(Y[i],Z[i,m]); }
```

$$Z_{i,m} = \begin{cases} 2i \leq m < 2(i+1) & \ldots \\ 2(i+1) \leq m \leq 3i & \ldots \end{cases}$$

$$Y_i = \max_{m=2i}^{3i} Z_{i,m}$$

Compound Expressions

- Consider the expression $A_{i,k} \otimes B_{k,j}$
  - The subexpression $A_{i,k}$ is shared along j
  - The subexpression $B_{k,j}$ is shared along i
- Combined expression has no sharing

Distributivity

- Consider a reduction of the form $\bigoplus_{j=1}^{i} A_{i,k} \otimes B_{k,j}$
- The combined expression has no sharing
- If $\otimes$ distributes over $\bigoplus$, we can do something
  - $A_{i,k}$ is constant within the summation
- We can replace the reduction with $A_{i,k} \otimes \bigoplus_{j=1}^{i} B_{k,j}$
- The reduction is over $B_{k,j}$ now
  - Has a non-trivial share space
  - Can decrease the complexity by one dimension

Final Example

- Consider the following: $Y_i = \sum_{j=1}^{i} \sum_{k=1}^{i} A_{i,j+k} \times B_{k,j}$
  - $A_{i,j+k}$ is shared along the vector (0,1,-1)
  - $B_{k,j}$ is shared along the vector (1,0,0)
- Combined expression has no sharing
- Can we exploit distributivity: Not yet!

Final Example

- Consider the following: $Y_i = \sum_{j=1}^{i} \sum_{k=1}^{i} A_{i,j+k} \times B_{k,j}$
- Reduction Decomposition may be applied

Final Example

- Consider the following: $Y_i = \sum_{j=1}^{i} \sum_{k=1}^{i} A_{i,j+k} \times B_{k,j}$
- Reduction Decomposition may be applied
  - The inner reduction has to be along the vector (0,1,-1)
  - Allows $A_{i,j+k}$ to be distributed out
  - $\sum B_{k,j}$ is the inner reduction. Has sharing
- Enables a one dimensional decrease in complexity

Applications

- RNA Secondary Structure Prediction
- Finite Element Methods

Message to YOU

- The Polyhedral Model is the granddaddy of many loop optimizations
  - Known for over 20 years
- Polyhedra are simple
- Can be used to solve reasonably important problems (ex. Decreasing complexity, others)

Previous Work

- Cocke and Keneddy (1977): "Strength Reduction"
- Bird (1984): "Program Derivation"
- Liu et. al. (2005): "Incrementalization"

Conclusions

- Characterization of complexity of reductions
- Mathematical foundations for simplification
- An algorithm that automatically and optimally applies different simplification transformations