# The Algebraic Path Problem

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**Abstract**

This is a brief intro to the algebraic path problem, and much of this material is based on Rote's excellent paper (Günter Rote: *A Systolic Array Algorithm for the Algebraic Path Problem (Shortest Paths; Matrix Inversion)*, Computing, 34(3):191–219, 1985)

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## 1 Introduction

The algebraic path problem (APP) unifies a number of well-known problems into a single algorithm schema. Warshall's transitive closure (TC) algorithm, Floyd's shortest path (SP) algorithm and the Gauss-Jordan matrix inversion (MI) algorithm are but instances of a single, generic algorithm for the APP (henceforth called the WFGJ algorithm), the only difference being the underlying algebraic structure. It's sequential complexity (and hence the work) is $\Theta(n^3)$ semiring operations.

The APP may be stated as follows. We are given a weighted graph $G = \langle V, E, w\rangle$ with vertices $V = \{1, 2, \ldots n\}$, edges $E \subseteq V \times V$ and a weight function $w : E \rightarrow \mathcal{S}$, where $\mathcal{S}$ is a semiring as defined below. A path in $G$ is a (possibly infinite) sequence of nodes $p = v_1 \ldots v_k$ and the weight of a path is defined as the product (in the semiring) $w(p) = w(v_1, v_2) \otimes w(v_2, v_3) \otimes \ldots \otimes w(v_{k-1}, v_k)$. For any set of paths $P$, we define $f(P) = \bigoplus_{p \in P} w(p)$ as the sum (again, as per the semiring definition) of the weights of all the paths in $P$.

We will first summarize the main properties of semirings that we use, and then develop the recurrences of the Warshall-Floyd-Gauss-Jordan algorithm.

## 2 Recap of semiring properties

In this section we briefly recap some of the semiring properties that are of concern here. This is not intended to be a complete discussion, but merely intended to make our presentation self contained.

**Definition 1.** *A* ***semiring****,* $\langle \mathcal{S}, \oplus, \otimes, \rangle$ *with zero* $\mathbf{0}$ *and unity* $\mathbf{1}$ *is an algebraic structure with two binary operations,* $\oplus$ *and* $\otimes$ *satisfying the following axioms.*

- $\mathbf{A_1}$ *$\langle \mathcal{S}, \oplus \rangle$ is a commutative semigroup with identity $\mathbf{0}$, called the* ***zero*** *of the semiring (i.e., $\mathcal{S}$ is closed under the associative and commutative operation $\oplus$, and $\mathbf{0}$ is the corresponding identity or neutral element).*

- $\mathbf{A_2}$ *$\langle \mathcal{S}, \otimes \rangle$ is a semigroup with identity $\mathbf{1}$, called the* ***unity*** *of the semiring.*

- $\mathbf{A_3}$ *$\otimes$ distributes over $\oplus$, i.e.,*

$$\begin{aligned} x \otimes (a \oplus b) &= (x \otimes a) \oplus (x \otimes b) \\ (a \oplus b) \otimes x &= (a \otimes x) \oplus (b \otimes x) \end{aligned}$$

- $\mathbf{A_4}$ *$\mathbf{0}$ is absorptive with respect to $\otimes$, i.e., $\mathbf{0} \otimes x = x \otimes \mathbf{0} = \mathbf{0}$*

*Two additional axioms are needed when dealing with infinite sums.*

- $\mathbf{A_5}$ *Let $I$ and $J$ be two countable (finite or infinite) sets. Then the following must hold whenever both sums on the right side are defined.*

$$\bigoplus_{(i,j) \in I \times J} a_i \otimes b_j = \bigoplus_{i \in I} a_i \otimes \bigoplus_{j \in J} b_j$$

- $\mathbf{A_6}$ *Let $I$, $K$, and $J$ be countable (finite or infinite) sets such that $\{J_k \mid k \in K\}$ is a partition of $I$. Then the following must hold whenever the left side is defined.*

$$\bigoplus_{i \in I} x_i = \bigoplus_{k \in K} \bigoplus_{i \in J_k} x_i$$

*Finally, we define a* ***closure*** *operation,* $*$ *(which may not always exist) for a particular kind of infinite sum:*

$$x* = \bigoplus_{i \geq 0} x^i = \mathbf{1} \oplus x \oplus (x \otimes x) \oplus (x \otimes x \otimes x) \oplus \dots$$

We now present four common examples.

**Example 1.** The most common example is the real numbers under conventional addition and multiplication, $\langle \mathcal{R}, +, . \rangle$, with zero, 0 and unity, 1. Indeed, this set is a field (it has additional properties that every element has an additive inverse, and every non-zero element admits a multiplicative inverse). Some of these additional properties will be necessary for defining the closure operation.

**Example 2.** Another example is $\langle \mathcal{R}', \min, + \rangle$, where $\mathcal{R}' = \mathcal{R} \cup \{\infty, -\infty\}$ is the set of reals, extended with plus and minus infinity, and assuming that $\infty + x = \infty$ and $(-\infty) + x = -\infty$ for all $x$, and that $(-\infty) + \infty = \infty$. The zero is $\infty$ and the unity is 0.

A well known variant is the so called **max-plus** algebra where min is replaced by max (and the zero is now $-\infty$).

**Example 3.** Another common example is the boolean semiring, $\langle \mathcal{B}, \vee, \wedge \rangle$, where $\mathcal{B} = \{0, 1\}$, with zero, 0 and unity, 1. Indeed, every bounded distributive lattice is a semiring.

**Example 4.** If $\langle \mathcal{S}, \oplus, \otimes \rangle$ is a semiring, then so also is $\langle \mathcal{M}^n_{\mathcal{S}}, +, \times \rangle$, where $\mathcal{M}^n_{\mathcal{S}}$ is the set of $n \times n$ matrices over $\mathcal{S}$, for any positive integer, $n$, and $+$ and $\times$ are matrix addition and multiplication. The zero and unity are the zero matrix and the identity matrix, respectively.

The closure operation for each of the above semirings is defined as follows (also note that for the reals, the closure operation is not always defined, and hence the APP as we will formulate later, may not always admit a solution).

- **Ex. 1** $x*$ may not always exist the following formula uses the field properties and operations $x* = \begin{cases} \frac{1}{1-x} & \text{if } x \neq 1 \\ \text{undefined} & \text{otherwise} \end{cases}$
- **Ex. 2** $x* = \begin{cases} 0 & \text{if } x \geq 0 \\ -\infty & \text{otherwise} \end{cases}$
- **Ex. 3** $x* = 1$
- **Ex. 4** For $n \times n$ matrices over a semiring too, $x*$ may not always exist. If subtraction is defined (i.e., $\oplus$ admits an inverse), $x*$ is the solution to $(\mathbf{I} - x)x* = \mathbf{I}$. Thus computing

$x*$ means inverting the matrix $\mathbf{I} - x$ (whenever possible) or equivalently, inverting $x$ means computing $(\mathbf{I} - x)*$.

## 3 The Warshall-Floyd-Gauss-Jordan Recurrences

Let $P(i,j)$ denote the (possibly infinite) set of all paths from $i$ to $j$. The APP is the problem of computing, for all pairs $i, j$, such that $0 < i, j \le n$, the value $d(i,j)$ (whenever it exists) defined as follows.

$$d(i,j) = f((P(i,j)) = \bigoplus_{p \in P(i,j)} w(p)$$

Let $P(i,j,k)$ be the set of all paths from $i$ to $j$ of length $k$, and thus $P(i,j) = \biguplus_{k \ge 0} P(i,j,k)$ (where $\uplus$ is disjoint union). Let also $A$ be the incidence matrix of $G$. Observe that $A^k = \bigoplus_{p \in P(i,j,k)} w(p)$. Hence, by axioms $\mathbf{A_5}$ and $\mathbf{A_6}$, $\mathbf{I} + A + A^2 \ldots$ is the matrix of the solution of the APP. Hence, solving the APP for a graph $G$ computes the closure of its incidence matrix $A$ (i.e., the inverse of $\mathbf{I} - A$).

Let $M_{i,j}^k$ denote the set of paths from $i$ to $j$, whose *intermediate* nodes are all in the set $\{1, \ldots k\}$, with the following conventions for paths with fewer than two edges. A path of length 1 (i.e., a single edge) has its intermediate nodes belonging to the null set—thus the edge from $i$ to $j$ is the only path in $M_{i,j}^0$. The intermediate nodes of a path of length 0 (i.e., the trivial path, $\epsilon_i$ from a node $i$ to itself which has no edge, and whose weight is $\mathbf{1}$, by convention) belong to the set $\{i\}$. Hence this path belongs to $M_{i,i}^i$ but not to $M_{i,i}^{i-1}$.

We define the sum of the weights of all paths in $M_{i,j}^k$ as

$$F(i,j,k) = f(M_{i,j}^k) = \bigoplus_{p \in M_{i,j}^k} w(p)$$

Since $M_{i,j}^n$ is the set of *all* paths from $i$ to $j$, we have $d(i,j) = F(i,j,n)$, and $F(i,j,0)$ is simply the weight of the edge from $i$ to $j$ (if it exists, otherwise $\mathbf{0}$).

The usual notion of concatenation of paths $p \circ q$ is extended to sets of paths: $P \circ Q = \{p \circ q \mid p \in P, q \in Q\}$. Hence, by axiom $\mathbf{A_5}$, $f(P \circ Q) = f(P) \otimes f(Q)$.

The recurrence relations for computing $F(i,j,k)$ are obtained as follows. We have the following three cases.

- The nodes, $i$ and $j$ are distinct from $k$, i.e., $i \neq k$; $j \neq k$. In this case, the set $M_{i,j}^k$ can be partitioned into two subsets, corresponding to paths that do not touch node $k$, and those that pass (at least once) through $k$.
  - The former is just $M_{i,j}^{k-1}$, and the sum of the weights of all the paths in it is $F(i,j,k-1)$.
  - All paths in the latter partition (say $M'$) can be uniquely decomposed into two parts: a prefix subpath from $i$ to the *last* visit to node $k$ and a suffix from $k$ to $j$, which does not have $k$ as an intermediate node. The set of all the prefixes is nothing but the set $M_{i,k}^k$ and the set of all suffixes is $M_{k,j}^{k-1}$. Conversely, any two paths from $M_{i,k}^k$ and $M_{k,j}^{k-1}$ can be concatenated to yield a path in $M'$. Thus, $M' = M_{i,k}^k \circ M_{k,j}^{k-1}$

  Hence, $M_{i,j}^k = M_{i,j}^{k-1} \uplus M_{i,k}^k \circ M_{k,j}^{k-1}$, and by axioms $\mathbf{A_5}$ and $\mathbf{A_6}$ we have

  $$F(i,j,k) = F(i,j,k-1) \oplus F(i,k,k) \otimes F(k,j,k-1)$$

- The nodes $i$ and $j$ are distinct but one of them is identical to $k$.
  - If $i = k$ we decompose any path in $M_{i,j}^i$ as above into a prefix from $M_{i,i}^i$, as above and a suffix from $M_{i,j}^{i-1}$. Conversely, any two paths from $M_{i,i}^i$ and $M_{i,j}^{i-1}$ can be composed together, yielding a path in $M_{i,j}^i$. Hence, $M_{i,j}^i = M_{i,i}^i \circ M_{i,j}^{i-1}$ and

    $$F(i,j,i) = F(i,i,i) \otimes F(i,j,i-1)$$

  - If $j = k$ we now decompose a path in $M_{i,j}^j$ into a prefix from $i$ to the *first* visit to $j$ and a suffix consisting of all subsequent cycles, if any from $j$ to itself. Thus $M_{i,j}^j = M_{i,j}^{j-1} \circ M_{j,j}^j$, and

    $$F(i,j,j) = F(i,j,j-1) \otimes F(j,j,j)$$

- Finally, if $i = j = k$, then any path in $M_{i,i}^i$, i.e., a cycle from $i$ to $i$ (with intermediate nodes in $\{1 \dots i\}$) is uniquely decomposable into a number of simple cycles, i.e., which do not have $i$ as an intermediate node, and hence which belong to $M^{i-1}_{i,i}$. Hence, $M^{i}_{i,i} = \{\epsilon_i\} \uplus M^{i-1}_{i,i} \uplus M^{i-1}_{i,i} \circ M^{i-1}_{i,i} \uplus M^{i-1}_{i,i} \circ M^{i-1}_{i,i} \circ M^{i-1}_{i,i} \uplus \ldots$ and

$$F(i,i,i) = F(i,i,i-1)*$$

Hence the WFGJ algorithm is specified by the following SRE, where $D = \{i,j,k \mid 1 \leq i,j \leq n; 0 \leq k \leq n\}$ is the domain of $F$.

$$d(i,j) = \{i,j \mid 1 \leq i,j \leq n\} : F(i,j,n) \tag{1}$$

$$F(i,j,k) = \begin{cases} D \cap \{i,j,k \mid k = 0\} & : a_{i,j} \\ D \cap \{i,j,k \mid i = j = k\} & : F(i,j,k-1)* \\ D \cap \{i,j,k \mid i = k \neq j\} & : F(k,k,k) \otimes F(i,j,k-1) \\ D \cap \{i,j,k \mid j = k \neq i\} & : F(i,j,k-1) \otimes F(k,k,k) \\ D \cap \{i,j,k \mid i \neq k; j \neq k; k > 0\} & : F(i,j,k-1) \oplus \\ & \quad (F(i,k,k) \otimes F(k,j,k-1)) \end{cases} \tag{2}$$