Analysis of an Important Class of Non-Markov Systems

Yashwant K. Malaiya, Member IEEE
State University of New York, Binghamton
Stephen Y.H. Su, Senior Member IEEE
State University of New York, Binghamton

Key Words—**Transient failures, Reliability analysis, Non-Markov processes.**

Reader Aids—
Purpose: Develop methods for analysis
Special math needed for explanation: Probability
Special math needed to use results: Same
Results useful to: Fault-tolerant computing, Reliability theoreticians

Summary & Conclusions—**Probabilistic modeling of many types of systems generally assumes Markov behavior. However, some important practical systems exhibit memory. For example, in digital computer systems, the probability of occurrence of a transient failure is related to the time period the system has been operating correctly. Analytic methods do not yet exist that allow accurate modeling of such systems for the purpose of reliability analysis and fault-tolerant design.**

Methods are presented here to analyze an important class of non-Markov systems. In this class, the transition-probability-rate of an outward transition from a state is related to the duration the system has continuously been in that state. To analyze such systems, concept of memory profile has been introduced. Methods are first presented which enable computation of steady-state probabilities for both discrete-time and continuous-time processes with two states. These are then extended for general non-steady-state cases and also for systems with more than two states.

I. INTRODUCTION

A wide range of systems has been modeled by Markov processes. For discrete-state continuous-time Markov process, the transition-probability-rate outwards from a state is governed by a constant parameter. This implies that the continuous-time duration spent in a state is exponentially distributed [1]. The advantage of assuming a process to be Markov is that the mathematical analysis is tractable [2-5]. The arrival of transient (intermittent) faults in digital circuits has so far been analyzed assuming a Markov process. However, in a study of multiple-processor systems [6], the Weibull distribution is found to have a better match with experimental data. This implies that in such systems, the transition-probability rate from a state depends on the time the system has spent in that state. The methods presented here allow analysis of this important class of non-Markov systems.

The motivation in developing the methods was to analyze transient failures in digital systems more accurately. The methods are very general, and are not restricted to Weibull (which is more general than exponential) distribution. They can be used to compute the reliability and the availability of digital systems with (or without) redundancy. They can also be used to design proper test experiments for digital systems with transient faults. Because of their generality, these methods can also be used for other systems which exhibit similar non-Markov behaviour.

To examine a system in which transition-probability-rate depends on the duration the system has been in that state, requires considering the history of each state. We shall first consider, in section II, discrete-time systems to develop the necessary concepts; continuous-time systems are then examined in section III. The results for the continuous-time systems are next extended for non-steady-state and then finally for multi-state cases.

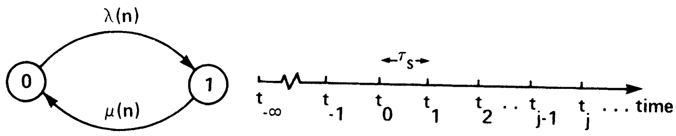

Fig. 1. Discrete time system with two states.

II. DISCRETE-TIME SYSTEMS

Consider a discrete-time system portrayed in figure 1.

Assumptions

1. It is a discrete-time 2-state system.
2. The parameter governing the transition out from a state depends on the time the system has spent in that state.
3. The system is in steady-state.

The steady-state probability of the system being in either state will be found here.

Notation

τ_s	time-step
$\lambda(n)$	Pr{system is in state 1 at instant t_j | (it was in state 0 at instant t_{j-1}) ∩ (system was in state 0 for a period $n\tau_s$)}
$\mu(n)$	Pr{system is in state 0 at instant t_j | (it was in state 1 at instant t_{j-1}) ∩ (system was in state 1 for a period $n\tau_s$)}
$P_x(j)$	Pr{system is in state x at instant t_j}; $x = 0$ or 1
$\pi_x(i, j)$	Pr{(system is in state x at instant t_j) ∩ (it entered state x $i\cdot\tau_s$ time ago)}; $x = 0$ or 1

The function $\pi_x(i, j)$, $i = 1$ to ∞ is called *memory-profile,* and describes the probabilistic history of state x, $x = 0$ or 1, at instant t_j. The first index i refers to the history (with the dimensions of time) and the second index j refers to the instant in time, when the memory profile is defined.

0018-9529/82/0400-0064$00.75

It was necessary to define $\pi_x(i, j)$ so that the effect of memory could be represented.

The sum of all the values of the memory-profile $\pi_x(i, j)$, $i = 1$ to ∞ is $p_x(t_j)$.

Proof: By definition we have—

$$p_x(j) = \Pr\{\text{system in the state } x \text{ at instant } t_j\}$$
$$= \sum_{i=1}^{\infty} \Pr\{\text{system in state } x \text{ at instant } t_j | \text{it entered state } x\ i{\cdot}\tau_s \text{ time ago}\}. \Pr\{\text{it entered state } x\ i{\cdot}\tau_s \text{ time ago}\}$$
$$= \sum_{i=1}^{\infty} \Pr\{(\text{system is in state } x \text{ at instant } t_j) \cap (\text{it entered state } x\ i{\cdot}\tau_s \text{ time ago})\}$$
$$= \sum_{i=1}^{\infty} \pi_x(i, j). \qquad QED \quad (1)$$

In order to obtain expressions for $p_x(j)$, $x = 0$ or 1, all $\pi_x(i, j)$ have to be evaluated. We first obtain an expression for $\pi_x(i, j)$, $i \geqslant 2$ in terms of $\pi_x(1, j)$. Then we evaluate $\pi_x(1, j)$ itself.

Consider for $i \geqslant 2$:

$$\pi_x(i, j) = \Pr\{\text{system is in state 0 with history } i{\cdot}\tau_s \text{ at time } t_{j+1}\}$$
$$= \Pr\{\text{system is in state 0 with history } (i-1){\cdot}\tau_s \text{ at } t_j\}$$
$$- \Pr\{\text{a transition to state 1 at time } t_j | \text{history is } (i-1){\cdot}\tau_s\}$$
$$= \pi_0(i-1, j) - \lambda(i-1)\, \pi_0(i-1, j)$$
$$= [1 - \lambda(i-1)]\, \pi_0(i-1, j) \qquad (2)$$

The solution to (2) is:

$$\pi_0(i, j) = \pi_0(i, j) \prod_{k=1}^{i-1} (1 - \lambda(k)),\ i \geqslant 2 \qquad (3a)$$

Similarly

$$\pi_1(i, j) = \pi_1(1, j) \prod_{k=1}^{i-1} (1 - \mu(k)),\ i \geqslant 2 \qquad (3b)$$

Now an expression will be obtained for $\pi_x(1, j)$, $x = 0$ or 1. Consider,

$$\pi_0(1, j) = \Pr\{\text{system is in state 0 with history } \tau_s \text{ at time } t_j\}$$
$$= \sum_{k=1}^{\infty} \Pr\{\text{a transition to state 0 at time } t_{j-1} | (\text{system in state 1 at } t_{j-1}) \cap (\text{it entered state 1 at } k\tau_s \text{ time ago})\} \cdot \Pr\{(\text{system in state 1 at } t_{j-1}) \cap (\text{it entered state 1 at } k\tau_s \text{ time ago})\}$$
$$= \sum_{k=1}^{\infty} \mu(k)\, \pi_1(k, j-1).$$

As in the steady-state case $\pi_1(k, j-1) = \pi_1(k, j)$,

$$\pi_0(1, j) = \sum_{k=1}^{\infty} \mu(k)\, \pi_1(k, j), \qquad (4a)$$

$$\pi_1(1, j) = \sum_{k=1}^{\infty} \lambda(k)\, \pi_0\text{–}(k, j). \qquad (4b)$$

We will next show that $\pi_0(1, j) = \pi_1(1, \text{j})$. Using (3b) and (4a), we have,

$$\pi_0(1, j) = \left\{\sum_{k=2}^{\infty} \mu(k) \cdot \pi_1(1, j) \prod_{k'=1}^{k-1} (1 - \mu(k'))\right\} + \mu(1)\, \pi_1(1, j)$$
$$= \pi_1(1, j) \left[\sum_{k=2}^{\infty} \mu(k) \prod_{k'=1}^{k-1} (1 - \mu(k')) + \mu(1)\right]$$
$$= \pi_1(1, j). \qquad (5)$$

The last step follows from the fact that the term within the brackets is equal to one as shown in the following proof.

Proof: Pr{there will be a transition to state 0 in infinite time | system was in state 1 at time 0 with history τ_s}

$$= \mu(1) + \mu(2)(1 - \mu(1)) + \mu(3)(1 - \mu(1))(1 - \mu(2)) + \ldots$$
$$= \mu(1) + \sum_{k=2}^{\infty} \mu(k) \prod_{k'=1}^{k-1} (1 - \mu)(k'))$$

However the l.h.s. is equal to one. Hence, the r.h.s is equal to one. *QED*

Since $p_0(j) + p_1(j) = 1$, using (1), (3), (5),

$$\pi_0(1, j)[\Lambda + M] = 1 \qquad (6)$$

$$\Lambda \equiv 1 + \sum_{i=2}^{\infty} \prod_{k=1}^{i-1} (1 - \lambda(k)) \qquad (7a)$$

$$M \equiv 1 + \sum_{i=2}^{\infty} \prod_{k=1}^{i-1} (1 - \mu(k)) \qquad (7b)$$

From (6) and (5)

$$\pi_0(1, j) = \pi_1(1, j) = 1/(\Lambda + M)$$

and hence

$$p_0(j) = \Lambda/(\Lambda + M),\ p_1(j) = M/(\Lambda + M) \qquad (8)$$

Once $\lambda(k)$ and $\mu(k)$ are given as functions of k, the steady-state probabilities $p_0(j)$ and $p_1(j)$ can be calculated using (8).

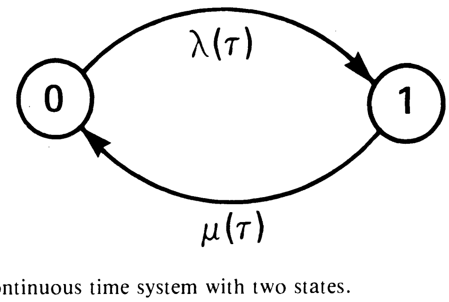

Fig. 2. Continuous time system with two states.

III. CONTINUOUS-TIME SYSTEMS

Continuous-time systems, represented in figure 2, can be similarly analyzed.

Assumptions

1. It is a continuous-time two-state system.
2. The parameter governing the transition out from a state depends on the time the system has spent in that state.
3. The system is in steady-state.

The steady-state probability of the system being in either state will be found here.

Notation

$\lambda(\tau)dt$ Pr{system is in state 1 at time t | (it was in state 0 at time $t - dt$) $\cap$ (system was in state 0 for a period τ)}
$\mu(\tau)dt$ Pr{system is in state 0 at time t | (it was in state 1 at time $t - dt$) $\cap$ (system was in state 1 for a period τ)}
$p_x(t)$ Pr{system is in state x at time t}; $x = 0$ or 1
$\pi_x(\tau, t)dt$ Pr{system is in state x at time t) $\cap$ (it entered state x between τ to $\tau + d\tau$ time ago}

The memory profile $\pi_x(\tau, t)$, $\tau = 0$ to ∞, is now continuous. For continuous cases, the $\lambda(\tau)$, $\mu(\tau)$, $\pi_0(\tau, t)$ all have dimensions of (time)$^{-1}$. Corresponding to (1), we now have—

$$p_x(t) = \int_0^\infty \pi_x(\tau, t)d\tau, x = 0, 1 \tag{9}$$

which means that $p_x(t)$ is the area under the memory-profile curve for time t.

For the steady-state case, we can get, corresponding to (2) and (3a)

$$\frac{\partial \pi_0(\tau, t)}{\partial \tau} = - \lambda(\tau)\pi_0(\tau, t), \tau > 0 \tag{10}$$

$$\pi_0(\tau, t) = \pi_0(0, t) \exp[-\int_0^\tau \lambda(\tau')d\tau'] \tag{11a}$$

Similarly,

$$\pi_1(\tau, t) = \pi_1(0, t) \exp[-\int_0^\tau \mu(\tau')d\tau'] \tag{11b}$$

Define—

$$\Lambda \equiv \int_0^\infty \exp[-\int_0^\tau \lambda(\tau')d\tau']d\tau \tag{12a}$$

$$M \equiv \int_0^\infty \exp[-\int_0^\tau \mu(\tau')d\tau']d\tau. \tag{12b}$$

Using (9) and (11),

$$p_0(t) = \pi_0(0, t)\Lambda \tag{13a}$$

$$p_1(t) = \pi_1(0, t)M \tag{13b}$$

Corresponding to (4), we now have—

$$\pi_0(0, t) = \int_0^\infty \pi_1(\tau', t)\mu(\tau')d\tau' \tag{14a}$$

$$\pi_1(0, t) = \int_0^\infty \pi_0(\tau', t)\lambda(\tau')d\tau'. \tag{14b}$$

The memory-profiles can be eliminated as in the previous section. The steady-state probabilities, corresponding to (8) are:

$$p_0(t) = \Lambda/(\Lambda + M), p_1(t) = M/(\Lambda + M). \tag{15}$$

When $\lambda(\tau)$ and $\mu(\tau)$ are given as a function of τ, (15) can be used to compute the steady-state probabilities. In the derivation above, $\lambda(\tau)$ and $\mu(\tau)$ are any general functions of τ. Two important special cases are examined below.

When the duration of staying in a state has an exponential distribution, then both λ and μ are constant with respect to history τ, and the system is described by a Markov process. In this case

$$p_0(t) = \mu/(\lambda + \mu) \text{ and } P_1(t) = \lambda/(\lambda + \mu) \tag{16}$$

which are well known results. For the Weibull distribution assume that:

$$\lambda(\tau) = \lambda'\alpha(\lambda'\tau)^{\alpha-1}$$

$$\mu(\tau) = \mu'\beta(\mu'\tau)^{\beta-1}$$

where α, β, λ', μ' are constant parameters. In this case;

$$p_0(t) = \mu'\Gamma(\frac{1+\alpha}{\alpha})/[\mu'\Gamma(\frac{1+\alpha}{\alpha}) + \lambda'\Gamma(\frac{1+\beta}{\beta})] \tag{17a}$$

$$p_1(t) = 1 - p_0(t) \tag{17b}$$

These can be easily verified by considering the average time spent in each state. Only for 2-state system is the average time spent in each state directly related to the steady-state probabilities.

Equations (16) or (17) are very useful for analyzing fault-tolerance with transient (intermittent) failures.

IV. EXTENSION TO NON-STEADY-STATE CASE

Assumptions

1. It is a 2-state continuous time system.
2. The parameter governing the transition out from a state depends on the time the system has spent in that state.
3. The system need not be in steady-state.

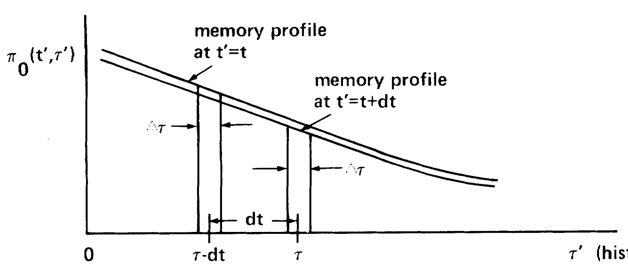

Fig. 3. Continuous memory profile for general case.

Consider the memory profiles at times t and $t + dt$, as shown in figure 3, these memory profiles might not be identical in general, as steady state is no longer assumed. Both $\Delta\tau$ and dt are infinitesimal, though not necessarily equal. Consider a small band of probability around history

τ-dt of width $\Delta\tau$ at a time t. The area enclosed in this band is equal to the probability that the system is in state 0 at time t, and it entered state 0 between $(t - dt)$-$(\Delta\tau)/2$ and $(\tau - dt) + (\Delta\tau)/2$ time ago. If an outgoing transition from state 0 were not allowed, then at time $t+dt$, a band of width $\Delta\tau$ around τ would represent the same probability. However, since transitions from state 0 are allowed, the band at τ, bounded by the memory profile at $t+dt$, would be shorter than the one at τ-dt bounded by the memory profile at t. The difference would be related to the probability rate of outgoing transitions from state 0.

$$[\pi_0(\tau, t + dt) - \pi_0(\tau - dt, t)]\Delta\tau = -\pi_0(\tau - dt, t)\Delta\tau\lambda(\tau - dt),$$

$$\text{for } \tau > 0$$

$$[\pi_0(\tau, t + dt) - \pi_0(\tau, t) + \pi_0(\tau, t) - \pi_0(\tau - dt, t)]$$

$$= -\pi_0(\tau - dt, t)\lambda(\tau - dt), \text{ for } \tau > 0$$

$$\pi_0(0, t)dt = dt \int_0^\infty \mu(\tau)\pi_1(\tau, t)d\tau, \text{ for } \tau = 0.$$

Following the usual calculus procedures, we have

$$\frac{\partial \pi_0(\tau, t)}{\partial t} = -\frac{\partial \pi_0(\tau, t)}{\partial \tau} - \pi_0(\tau, t)\lambda(\tau), \text{ for } \tau > 0 \quad (18a)$$

$$\frac{\partial \pi_0(0, t)}{\partial t} = \int_0^\infty \mu(\tau) \frac{\partial \pi_1(\tau, t)}{\partial t} d\tau, \text{ for } \tau = 0. \quad (18b)$$

The steady state case can be obtained from (18a) by taking

$$\frac{\partial \pi_0(\tau, t)}{\partial t} = 0.$$

$$\text{i.e. } \frac{\partial \pi_0(\tau, t)}{\partial \tau} = -\pi_0(\tau, t)\lambda(\tau), \text{ for } \tau > 0$$

which is identical to (10). For steady-state in (18b), a similar procedure yields

$$\frac{\partial \pi_1(\tau, t)}{\partial t} = 0.$$

The system probabilities are completely defined by (18) for each state, including the initial conditions. There is some redundancy in information if an additional equation is added to indicate that the total probability of being in either state is unity.

V. EXTENSION TO MULTIPLE STATES

Equation (18) can be extended to more than two states.

Assumptions

1. It is a multiple-state continuous time system.
2. The parameters governing the transition out from a state depends on the time the system has spent in that state.
3. The system need not be in steady-state.

Let a transition from state k to state 1 be governed by the parameter $\lambda_{k1}(\tau)$. Then corresponding to (18), we have

$$\frac{\partial \pi_i(\tau, t)}{\partial t} = -\frac{\partial \pi_i(\tau, t)}{\partial \tau} - \pi_i(\tau, t) \sum_j \lambda_{ij}(\tau), \text{ for } \tau > 0 \quad (19a)$$

$$\pi_i(0, t) = \sum_{j \neq i} \int_0^\infty \lambda_{ji}(\tau)\pi_j(\tau, t)d\tau, \text{ for } \tau = 0. \quad (19b)$$

For the steady-state case,

$$\frac{\partial \pi_i(\tau, t)}{\partial \tau} \doteq -\pi_i(\tau, t) \sum_j \lambda_{ij}(\tau), \text{ for } \tau > 0$$

$$\pi_i(\tau, t) = \pi_i(0, t) \exp[-\int_0^\tau \sum_j \lambda_{ij}(\tau')d\tau'], \text{ for all } i \quad (20a)$$

Using this in (19b) yields

$$\pi_i(0, t) = \sum_{j \neq i} \pi_j(0, t) \int_0^\infty \lambda_{ji}(\tau)$$

$$\exp[-\int_0^\tau \sum_k \lambda_{jk}(\tau')d\tau']d\tau, \text{ for all } i \quad (20b)$$

Equation (20b) is a simple system of linear algebraic equations in $\pi_i(0, t)$, for all i. Once all $\pi_i(0, t)$ are obtained, $\pi_i(\tau, t)$ can be obtained by (20a).

ACKNOWLEDGMENT

We thank Mr. Chi-Chang Liaw for reviewing the paper during the development of this work. This work was supported by the Division of Mathematical and Computer Science of the National Science Foundation under Grant Number MCS78-24323.

REFERENCES

[1] M.L. Shooman, *Probabilistic Reliability,* McGraw-Hill, New York, 1968.

[2] S.Y.H. Su, I. Koren, Y.K. Malaiya, "A continuous parameter Markov model and detection procedures for intermittent faults", *IEEE Trans. Computers,* vol C-27, 1978 Jun, pp 567-570.

[3] Y.K. Malaiya, S.Y.H. Su, "A survey of methods for intermittent fault analysis", *Proc. National Computer Conf.,* 1979, pp 577-585.

[4] J. Savir, "Model and random-testing properties of intermittent faults in combinational circuits", *J. Design Automation and Fault-tolerant Computing,* vol 2, 1978 Jul, pp 215-229.

[5] I. Koren, S.Y.H. Su, "Reliability analysis of *N*-modular redundancy systems with intermittent and permanent faults", *IEEE Trans. Computers,* vol C-28, 1979 Jul, pp 514-520.

[6] S.R. McConnel, D.D. Siewiorek, M.M. Tsao, "The measurement and analysis of transient errors in digital computer systems", *Proc. FTCS-9,* Madison, 1979 Jun, pp 67-70.

[7] E. Parzen, *Stochastic Processes,* Holden-Day, New York, 1962.

AUTHORS

Dr. Yashwant K. Malaiya; Department of Computer Science; School of Advanced Technology; State University of New York; Binghamton, New York 13901 USA.

Yashwant Malaiya (S'76, S'78, M'78) was born in 1951 in Saugor, India. He received his BS in 1969, MS in Physics in 1971, both from the University of Saugor. He obtained the MS (Tech.) in electronics from BITS, Pilani in 1974 and his PhD in electrical engineering from Utah State University in 1978. He is an assistant professor in the Department of Computer Science. His interests are intermittent faults, reliability analysis, fault-tolerance in multiprocessor systems, and design automation.

Prof. Stephen Y.H. Su; Department of Computer Science; School of Advanced Technology; State University of New York; Binghamton, New York 13901 USA.

Stephen Su (S'65, M'67, SM'77) was born in 1938 in Fukien, China. He received his BS in electrical engineering from the National Taiwan University in 1960, and the MS in electrical engineering and his PhD degree in computer engineering both from the University of Wisconsin, Madison in 1963 and 1967, respectively. He has been a chairman of two international symposia and is a senior member of IEEE. He is a professor in the Department of Computer Science. His interests are intermittent faults, short-circuit and delay faults, design automation, and fault-tolerant design.

Manuscript TR80-74 received 1980 June 26; revised 1981 April 27; 1981 July 15.

★★★

FROM THE EDITORS

Practical Papers

We, the Editors, want to publish practical papers just as badly as you, the readers, want to see them. They are hard to find. We established the Special Papers Board to search out practical papers. We reject very few practical papers, because we want to encourage their submittal.

When next you complain about how few practical papers appear in this, or any similar journal, ask yourself this question: "When was the most recent time that I, or someone from this division of my company, submitted a practical paper to this journal?"

The main reason why you see so many theoretical papers, and so few practical papers, is that university professors get paid to write theoretical papers, and engineers get paid to produce hardware rather than to write papers about it. Both do their jobs reasonably well.

Examples of general areas which both you and we want are as follows.

1. Case histories.
2. Reliability techniques which engineers actually found to be useful on a job, and those which were not useful or were too expensive.
3. How engineers and managers set realistic R&M requirements for a system or equipment.
4. The kinds of reliability testing which were actually cost-effective.
5. Use of reliability data from the field instead of from special reliability tests. Comparison of field data with reliability tests.
6. Engineering comments on the worth of standards such as the international or US military standards on R&M.
7. Ideas, from experience, on the major obstacles to setting and achieving worthwhile reliability requirements in commercial, military, or other fields.
8. Where to find information. For example, a list of trade and professional journals of value to reliability and quality control practitioners.
9. Information summaries. For example, annotated lists of computer programs for analyzing electronic circuits or for generating fault trees; tell what the programs do, how big a computer they need, and where they are available.
10. Papers without equations.
11. Actual R&M field experience with several technologies for accomplishing the same thing; e.g., in the microwave field.

If you have an idea for a practical paper you would like to write, let us know. The Editor and our Special Papers Board will help you with it. Or give us the name of a colleague who wants to write a practical paper.

Extended Abstracts

Scope. This policy applies to a paper which appears to be a mathematical exercise of well-known principles without providing an insight or tool for a manager, engineer, scientist, or theoretician. Such a paper is not published in full.

The application of this policy is shifting so that the burden of proof lies largely on the paper itself to show suitability, rather than on the Editor to show unsuitability.

Effect. If the Editor, with the assistance of the referees, judges that an otherwise suitable paper falls under the scope of this policy, then only an Extended Abstract is published in this Transactions. The full paper is placed in a Supplement.

Extended Abstract. The maximum length is one manuscript page (including everything).

Only one author is listed. The mailing address, but no biography, is given.

No symbols, no equations, no references (except to the Supplement itself), nor any figures are allowed.

The title and author are not listed in the Table of Contents of the issue.

All such papers are grouped in a special section labeled: Extended Abstracts. The Extended Abstracts section has its own Table of Contents placed at the front of that section.

Supplement. The entire paper is placed in a separately available Supplement. The complete set of authors with biographies can be placed there. The format is that of a traditionally published paper.

Implementation. It will take all of the 1981 calendar year for the policy to be completely implemented in the printed Transactions because of previous commitments. The Editor will attempt to publish all such old papers before publishing the first Extended Abstract section.

★★★