CS 301 - Lecture 6
Nonregular Languages and the Pumping Lemma

Fall 2008

Review

- Languages and Grammars
  - Alphabets, strings, languages
- Regular Languages
  - Deterministic Finite Automata
  - Nondeterministic Finite Automata
  - Equivalence of NFA and DFA
  - Minimizing a DFA
  - Regular Expressions
  - Regular Grammars
  - Properties of Regular Languages
- Today:
  - Properties of Regular Languages
  - Pumping lemma for regular languages

We say: Regular languages are **closed under**

Union: $L_1 \cup L_2$

Concatenation: $L_1 L_2$

Star: $L_1 *$

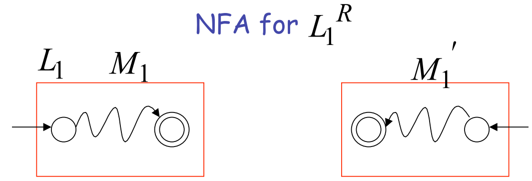
Reversal: $L_1{}^R$

Complement: $\overline{L_1}$

Intersection: $L_1 \cap L_2$

Reverse

NFA for $L_1{}^R$



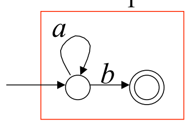
1. Reverse all transitions

2. Make initial state final state and vice versa

Example

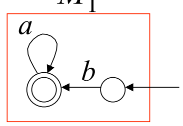
M_1

$L_1 = \{a^n b\}$



M_1'

$L_1^R = \{ba^n\}$



Intersection

DeMorgan's Law: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

$L_1\ ,\ L_2$ regular

$\Rightarrow$ $\overline{L_1}\ ,\ \overline{L_2}$ regular

$\Rightarrow$ $\overline{L_1} \cup \overline{L_2}$ regular

$\Rightarrow$ $\overline{\overline{L_1} \cup \overline{L_2}}$ regular

$\Rightarrow$ $L_1 \cap L_2$ regular

Example

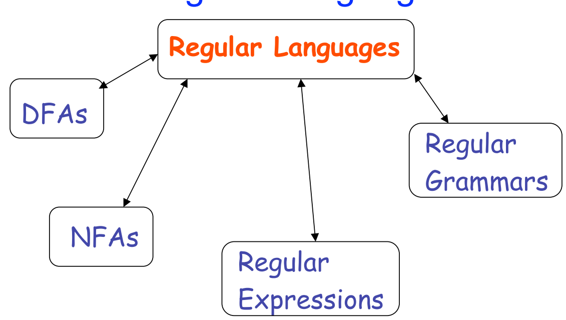
$L_1 = \{a^n b\}$ regular

$L_2 = \{ab, ba\}$ regular

$\Rightarrow$ $L_1 \cap L_2 = \{ab\}$

regular

Standard Representations of Regular Languages



When we say: We are given a Regular Language L

We mean: Language L is in a standard representation

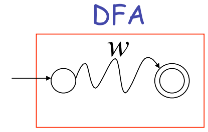
Elementary Questions about Regular Languages

Membership Question

Question: Given regular language L and string w how can we check if $w \in L$?

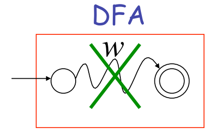
Answer: Take the DFA that accepts L and check if w is accepted

DFA



$w \in L$

DFA



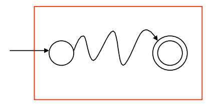
$w \notin L$

Question: Given regular language L
how can we check
if L is empty: $(L = \emptyset)$?

Answer: Take the DFA that accepts L

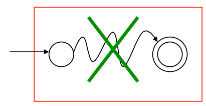
Check if there is any path from
the initial state to a final state

DFA



$L \neq \emptyset$

DFA



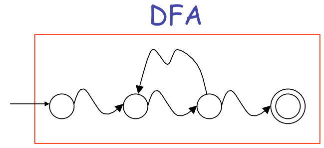
$L = \emptyset$

Question: Given regular language L
how can we check
if L is finite?

Answer: Take the DFA that accepts L

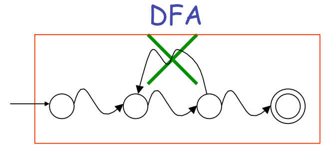
Check if there is a walk with cycle
from the initial state to a final state

DFA



L is infinite

DFA



L is finite

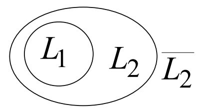
Question: Given regular languages L_1 and L_2
how can we check if $L_1 = L_2$?

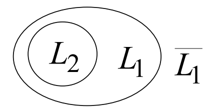
Answer: Find if $(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \emptyset$

$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) = \emptyset$$

$$\Downarrow$$

$L_1 \cap \overline{L_2} = \emptyset$ and $\overline{L_1} \cap L_2 = \emptyset$





$L_1 \subseteq L_2$ $\quad$ $L_2 \subseteq L_1$

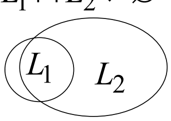
$$\Downarrow$$

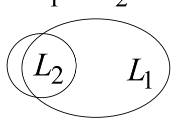
$$L_1 = L_2$$

$$(L_1 \cap \overline{L_2}) \cup (\overline{L_1} \cap L_2) \neq \emptyset$$

$$\Downarrow$$

$L_1 \cap \overline{L_2} \neq \emptyset$ or $\overline{L_1} \cap L_2 \neq \emptyset$





$L_1 \not\subseteq L_2$ $\quad$ $L_2 \not\subseteq L_1$

$$\Downarrow$$

$$L_1 \neq L_2$$

When we say: We are given
a Regular Language L

We mean: Language L is in a standard
representation

Non-regular languages

$\{a^n b^n : \ n \geq 0\}$

$\{vv^R : \ v \in \{a,b\}^*\}$

Regular languages

a^*b

$b^*c + a$

$b + c(a+b)^*$

$etc...$

How can we prove that a language L is not regular?

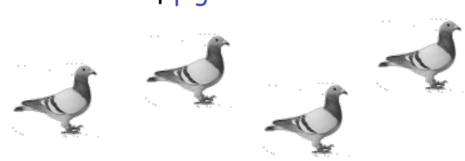
Prove that there is no DFA that accepts L

Problem: this is not easy to prove

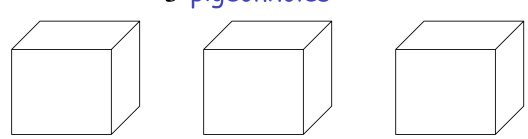
Solution: the Pumping Lemma !!!

The Pigeonhole Principle

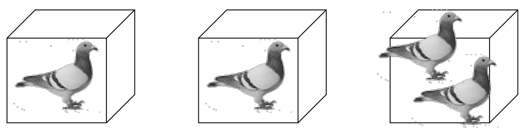
4 pigeons



3 pigeonholes

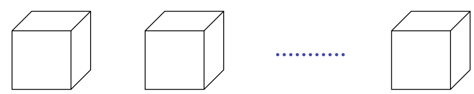


A pigeonhole must contain at least two pigeons



n pigeons

m pigeonholes $n > m$



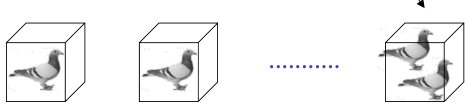
The Pigeonhole Principle

n pigeons

m pigeonholes

$n > m$

There is a pigeonhole with at least 2 pigeons

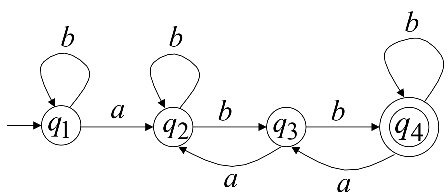


The Pigeonhole Principle

and

DFAs

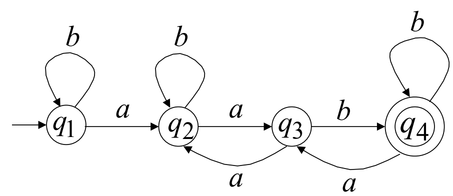
DFA with 4 states



In walks of strings: a no state is repeated

aa

aab

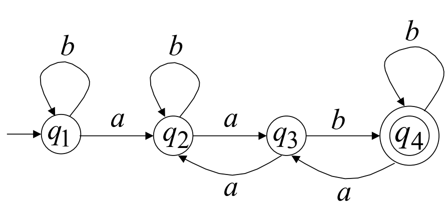


In walks of strings: $aabb$ a state is repeated

$bbaa$

$abbabb$

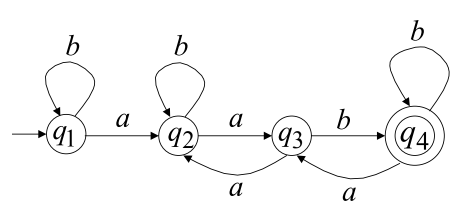
$abbbabbabb...$



If string w has length $|w| \geq 4$:

Then the transitions of string w are more than the states of the DFA

Thus, a state must be repeated

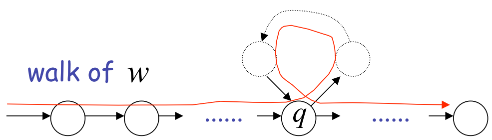


In general, for any DFA:

String w has length $\geq$ number of states

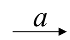
⇓

A state q must be repeated in the walk of w



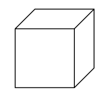
Repeated state

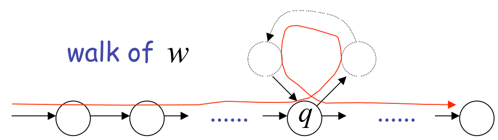
In other words for a string w:



transitions are pigeons

states are pigeonholes



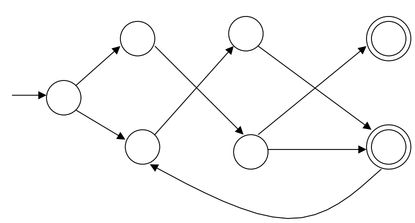


Repeated state

The Pumping Lemma

Take an infinite regular language L

There exists a DFA that accepts L



m
states

Take string w with $w \in L$

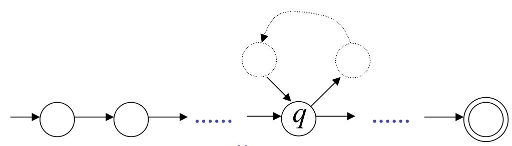
There is a walk with label w:

walk w

If string w has length $|w| \geq m$ (number of states of DFA)

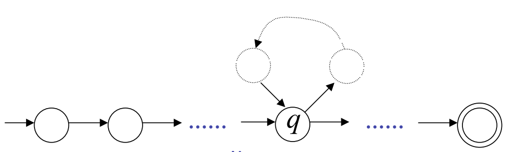
then, from the pigeonhole principle:

a state is repeated in the walk w



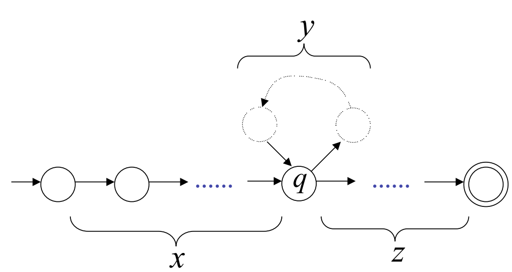
walk w

Let q be the first state repeated in the walk of w



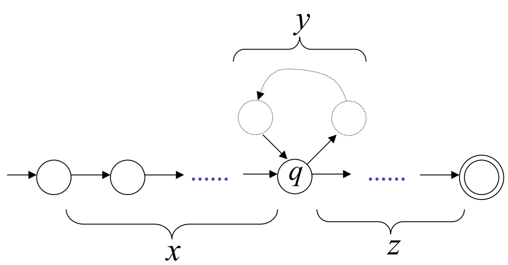
walk w

Write $w = x\ y\ z$

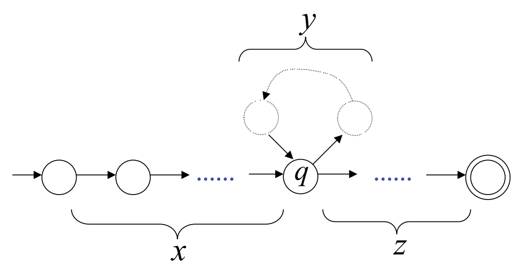


Observations: length $|x\,y| \le m$ number of states of DFA

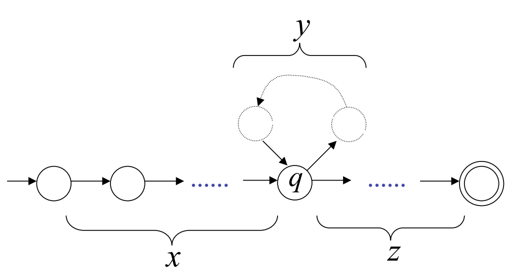
length $|y| \ge 1$



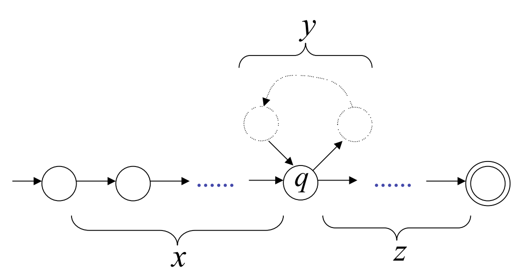
Observation: The string $x\,z$ is accepted



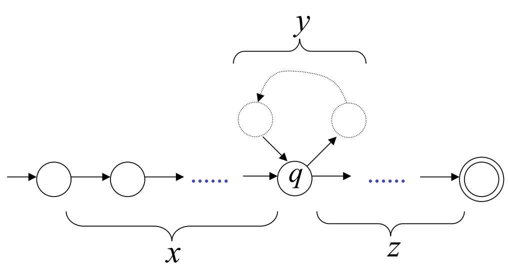
Observation: The string $x\,y\,y\,z$ is accepted



Observation: The string $x\,y\,y\,y\,z$ is accepted

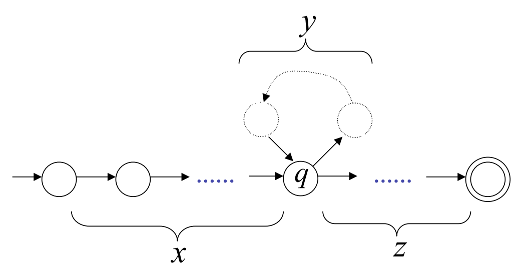


In General: The string $x\,y^i\,z$ is accepted $i = 0, 1, 2, \ldots$



In General: $xy^iz \in L$ $i = 0, 1, 2, \ldots$

Language accepted by the DFA



In other words, we described:

The Pumping Lemma !!!

The Pumping Lemma:

- Given a infinite regular language L
- there exists an integer m
- for any string $w \in L$ with length $|w| \geq m$
- we can write $w = x\,y\,z$
- with $|x\,y| \leq m$ and $|y| \geq 1$
- such that: $x\,y^i\,z \in L$ $i = 0, 1, 2, \ldots$

Applications of the Pumping Lemma

Theorem: The language $L = \{a^n b^n : n \geq 0\}$ is not regular

Proof: Use the Pumping Lemma

$L = \{a^n b^n : n \geq 0\}$

Assume for contradiction that L is a regular language

Since L is infinite we can apply the Pumping Lemma

$L = \{a^n b^n : n \geq 0\}$

Let m be the integer in the Pumping Lemma

Pick a string w such that: $w \in L$

length $|w| \geq m$

We pick $w = a^m b^m$

Write: $a^m b^m = x\,y\,z$

From the Pumping Lemma
it must be that length $|x\,y| \le m, \quad |y| \ge 1$

$$xyz = a^m b^m = \overbrace{a...aa...aa...a}^{m}\overbrace{b...b}^{m} = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...ab...b}_{z}$$

Thus: $y = a^k, \quad k \ge 1$

$x\,y\,z = a^m b^m \qquad y = a^k, \quad k \ge 1$

From the Pumping Lemma: $x\,y^i\,z \in L$

$i = 0, 1, 2, \ldots$

Thus: $x\,y^2\,z \in L$

$x\,y\,z = a^m b^m \qquad y = a^k, \quad k \ge 1$

From the Pumping Lemma: $x\,y^2\,z \in L$

$$xy^2z = \overbrace{a...aa...aa...aa...a}^{m+k}\overbrace{b...b}^{m} = \underbrace{a...a}_{x}\underbrace{a...a}_{y}\underbrace{a...a}_{y}\underbrace{a...ab...b}_{z} \in L$$

Thus: $a^{m+k} b^m \in L$

$a^{m+k} b^m \in L$

BUT: $L = \{a^n b^n : n \ge 0\}$

$\Downarrow$

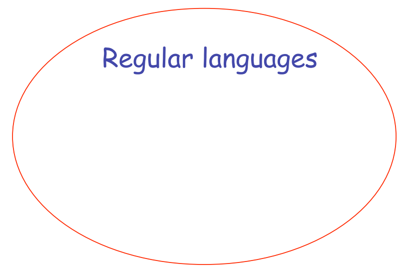
$a^{m+k} b^m \notin L$

CONTRADICTION!!!

Therefore: Our assumption that L is a regular language is not true

Conclusion: L is not a regular language

Non-regular languages $\{a^n b^n : n \geq 0\}$



What's Next

- Read
  - Linz Chapter 1, 2.1, 2.2, 2.3, (skip 2.4) 3, and 4
  - JFLAP Startup, Chapter 1, 2.1, 3, 4, 6.1
- Next Lecture Topics from Chapter 4.3
  - More Pumping Lemma
- Quiz 1 in Recitation on Wednesday 9/17
  - Covers Linz 1.1, 1.2, 2.1, 2.2, 2.3 and JFLAP 1, 2.1
  - Closed book, but you may bring one sheet of 8.5 x 11 inch paper with any notes you like.
  - Quiz will take the full hour
- Homework
  - Homework Due Today
  - New Homework Assigned Friday Morning
  - New Homework Due Following Thursday