Min-Orderable Digraphs

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Abstract

We unify several seemingly different graph and digraph classes under one umbrella. These classes are all, broadly speaking, different generalizations of interval graphs, and include, in addition to interval graphs, adjusted interval digraphs, complements of threshold tolerance graphs (known as co-TT graphs), bipartite interval containment graphs, bipartite co-circular arc graphs, and two-directional orthogonal ray bigraphs. (The last three classes coincide, but have been investigated in different contexts.) We show that all of the above classes are united by a common ordering characterization, the existence of a min ordering. However, because the presence or absence of reflexive relationships (loops) affect whether a graph or digraph has a min ordering, to obtain this result, we must define the graphs and digraphs to have those loops that are implied by their definitions. These have been largely ignored in previous work. We propose a common generalization of all these graph and digraph classes, namely *signed-interval digraphs*, characterized by the existence of a compact representation, a *signed-interval model*, which is a generalization of known representations of the graph classes. We show that the signed-interval digraphs are precisely those digraphs that are characterized by the existence of a min ordering when the loops implied by the model are considered part of the graph. We also offer an alternative geometric characterization of these digraphs. We show that co-TT graphs are the symmetric signed-interval digraphs, the adjusted interval digraphs are the reflexive signed-interval digraphs, and the interval graphs are the intersection of these two classes, namely, the reflexive and symmetric signed-interval digraphs. Similar results hold for bipartite interval containment graphs, bipartite co-circular arc graphs, and two-directional orthogonal ray bigraphs. [1]

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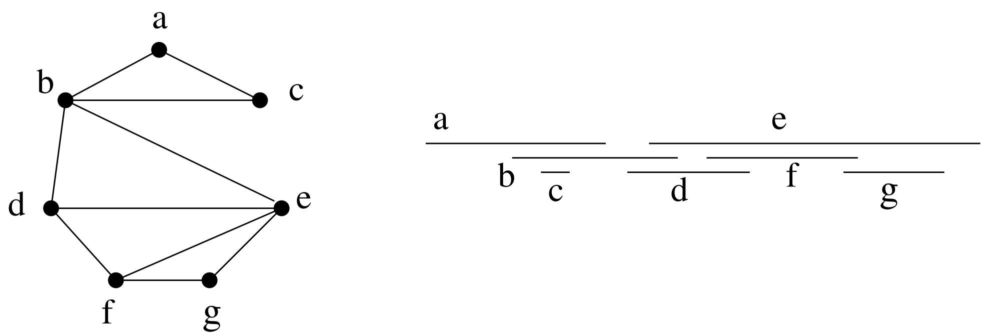

Figure 1: An interval graph and corresponding interval model. There is an implicit loop at each vertex.

1 Introduction

A digraph H is *reflexive* if each $vv \in E(H), v \in V(H)$ (every vertex in H has a loop); *irreflexive* if no $vv \in E(H)$ (no vertex in H has a loop); and *symmetric* if $ab \in E(H)$ implies $ba \in E(H)$. In this paper, we shall consider both graphs and digraphs; we view graphs as symmetric digraphs, by replacing each edge uv by the two arcs uv, vu. (In particular, *graphs* can have loops, and *irreflexive graphs* are loopless. We do not consider multiple edges.) However, in certain situations we view *bigraphs*, i.e., bipartite graphs with a fixed bipartition, as oriented graphs, with all edges oriented from one part of the bipartition to the other.

A graph H is an *interval graph* if it is the intersection graph of a family of intervals on the real line, i.e., if there exists a family of intervals $\{[x_v, y_v] | v \in V(H)\}$ such that $uv \in E(H)$ if and only if $[x_u, y_u] \cap [x_v, y_v] \neq \emptyset$. The family of intervals is an *interval model* of H. (See Figure 1.) Similarly, a graph is a *circular-arc graph* if it is the intersection graph of a family of arcs on the circle.

A graph H is a *threshold tolerance graph* [34] if each vertex v can be assigned a *weight* w_v and a *tolerance* t_v so that ab is an edge of H if and only if $w_a + w_b > \min(t_a, t_b)$. (When all t_v are equal, this defines a better known class of *threshold graphs* [6].) Those graphs that are the complements of threshold tolerance graphs, the *co-threshold tolerance graphs ("co-TT" graphs)* have also been shown to be those graphs that are representable with a generalization of an interval model, called a *co-TT model*. Details are given in the next section.

A generalization of interval models to directed graphs is the class of *adjusted-interval digraphs* [13], where each vertex has a *source interval* and a *sink interval* that share a common left endpoint, and for two vertices x and y, xy is a directed edge if the source interval of x intersects the sink interval of y. We discuss the model in more detail in the next section; an illustration is given in Figure 4. An interval model can be seen as the special case where the source interval for each vertex is equal to the sink interval for that vertex, necessitating only one interval to represent both.

A *submatrix* of a matrix M is the result of deleting any set of rows and columns of M, leaving the relative order of the remaining rows and columns intact. Henceforth, we

$$
\begin{array}{cc} & \begin{array}{cc} b' & b \end{array} \\ \begin{array}{c} a \\ a' \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}
\qquad
\begin{array}{cc} & \begin{array}{cc} b' & b \end{array} \\ \begin{array}{c} a \\ a' \end{array} & \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \end{array}
\qquad
\begin{array}{cc} & \begin{array}{cc} b' & b \end{array} \\ \begin{array}{c} a \\ a' \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}
$$

Figure 2: A min ordering of a digraph is an ordering of its vertices such that neither of the left two depicted submatrices Σ (left-most matrix) and Λ (middle matrix) occurs in the corresponding adjacency matrix; on the right, for comparison, is the matrix Γ.

will let Σ denote the matrix whose rows are 01 and 10 and let Λ denote the matrix whose rows are 01 and 11 (See Figure 2). We note that these names are arbitrary, as we were unfortunately not able to choose names that corresponded to the shape of the matrix, as is the case for Γ. Let M, A, and B be matrices. M is *A-free* if A is not the submatrix of M induced by any subset of its rows and columns, and it is $\{A, B\}$-free if it is A-free and B-free. A *min ordering* of a digraph H is a linear ordering $<$ of the vertices of H, so that $ab \in E(H), a'b' \in E(H)$ and $a < a', b' < b$ implies that $ab' \in E(H)$ [13] (cf. also [19], where min ordering is called an *X-underbar enumeration*).

In other words, a min ordering is an ordering of the vertices such that when the ordering of rows and columns of the adjacency matrix matches this ordering, it is $\{\Sigma, \Lambda\}$-free.

We note that the matrix with rows 11 and 10, called Γ in the literature [8, 32], is obtained from the matrix Λ by reversing the order of both rows and columns. (See Figure 2.) It follows that a matrix has a Γ-free ordering if and only if it has a Λ-free ordering. (Each is the reverse ordering of the other.) We include both options to be able to match the standard terminology in two different areas.

The presence or absence of loops (1's on the diagonal of the adjacency matrix) can affect whether the graph has a min ordering. It was pointed out in [13] that when loops are added to every vertex of an interval graph, it has a $\{\Sigma, \Lambda\}$-free ordering. (Equivalently, its *augmented adjacency matrix* has a min ordering.) In other words, we consider interval graphs to be reflexive. This quite naturally corresponds to the definition of interval graphs, since each interval intersects itself. Similarly, the model of adjusted interval digraphs implies that they are reflexive, since a vertex's source interval intersects its sink interval at their shared left endpoint.

In this paper, we observe that a co-TT model of a co-TT graph implies that some vertices have loops and others do not. This issue has been ignored in the previous literature on the class. In the present paper, we show that when the loops that are implied by a co-TT model of the graph are included, it is min-orderable. Although an ordering characterization, essentially equivalent to ours, was known [34], its relationship to min orderings has not been previously recognized. By explicitly considering the loops, we were able to view the ordering as a min-ordering, and thereby link co-TT graphs to the other classes having a min ordering.

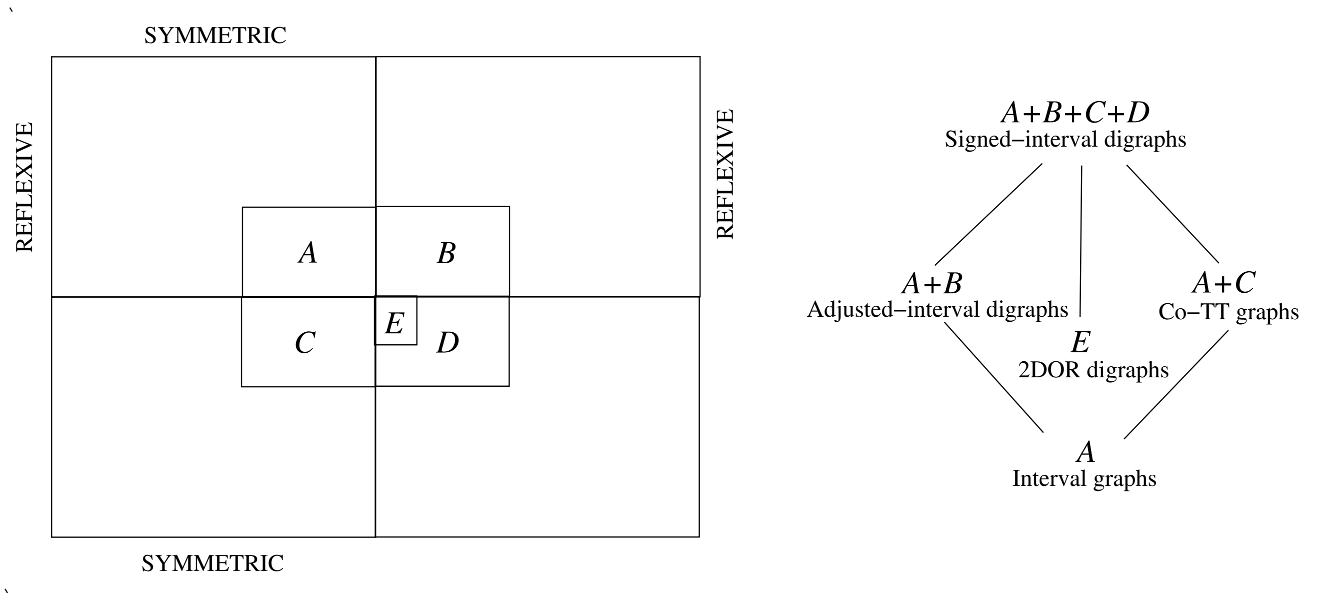

Figure 3: The world of digraphs.

The main goal of this paper is to promote a common generalization of all of these classes by combining elements of adjusted interval models and co-TT models, to obtain what we will call a *signed-interval model* of a digraph. We call the class of graphs that are representable with a signed-interval model the *signed-interval digraphs*. We note that we hyphenate the term *signed-interval* digraph in order to emphasize that it is the intervals that are signed (positive or negative), as distinguished from signed graphs or digraphs. The signed-interval model implies which vertices have loops and which do not. We show that when the implied loops are included in the digraph, it has a min ordering. We show that class of signed-interval digraphs is *equal* to the class of digraphs that have a min ordering, giving a characterization of the min-orderable digraphs in terms of representability with a signed-interval model.

When we view interval graphs and co-TT graphs as digraphs, we consider them to be symmetric digraphs, i.e., each edge uv is replaced by the two opposite arcs uv and vu. With this in mind, the classes of interval graphs, co-TT graphs, and adjusted interval digraphs, are all subclasses of the class of signed-interval digraphs. We show that in fact interval graphs are exactly the subclass of signed-interval digraphs that are symmetric and reflexive, the co-TT graphs are the subclass that are symmetric, and the adjusted interval digraphs are the subclass that are reflexive.

As mentioned earlier, we view bigraphs differently. A *uniform orientation* of bipartite graph G is the digraph that results from selecting a bipartition$\{A, B\}$ of G and orienting all of its edges from A to B. (The choice of which bipartition is taken is arbitrary.) Note that the uniform orientations of bipartite graphs are precisely those irreflexive digraphs where every vertex is a source or a sink. We will show that a uniform orientation of a bigraph G is a signed-interval digraph if and only if G is the complement of a circular-arc graph.

It follows from [11, 25, 29, 36] that the class of bipartite graphs that are complements of circular-arc graphs is equal to the class of *interval containment bigraphs* and is also equal to the class of *two-directional orthogonal-ray bigraphs*, defined below. We will call uniform orientations of these bigraphs *two-directional orthogonal-ray digraphs* or *2DOR digraphs*, cf. Figure 3, remembering that the class has all these equivalent descriptions. We emphasize once more that while interval graphs and co-TT graphs are viewed as symmetric digraphs, these bigraphs (two-directional orthogonal-ray bigraphs, interval containment bigraphs, and bipartite graphs that are complements of circular-arc graphs) are viewed as uniform orientations of bipartite graphs.

Because the uniform orientations of these bipartite graphs are irreflexive, their uniform orientations are disjoint from the adjusted interval digraphs, hence disjoint from the interval graphs. Because they are antisymmetric, their intersection with the co-TT graphs is trivial: it is the class of edgeless, loopless digraphs, the only loopless digraphs that are both symmetric and antisymmetric.

In Figure 3 we illustrate these relationships on a grid representing all digraphs, with the top half being reflexive digraphs and the left half being symmetric digraphs. The central rectangle, the region $A \cup B \cup C \cup D$, represents the class of signed-interval digraphs (i.e., min-orderable digraphs), while $A \cup B$ and $A \cup C$ represent adjusted-interval digraphs and co-TT graphs, respectively. The small rectangle E represents the uniform orientations of two-directional orthogonal-ray bigraphs. (It has a trivial intersection with the region C, not shown in the figure, as noted in the preceding paragraph.)

The lower half of the large rectangle corresponds to all digraphs that are not reflexive; the small rectangle E belongs to the region of irreflexive digraphs (not marked in the figure). Similarly, the right half of the larger rectangle corresponds to all digraphs that are not symmetric, while E lies in the region of antisymmetric digraphs (unmarked in the figure). In fact, E is the intersection of the class of uniform bipartite digraphs (which are irreflexive and antisymmetric) and the rectangle $A \cup B \cup C \cup D$.

A graph G is *chordal* if every cycle C of length greater than three in G has a *chord*, i.e., a non-loop edge not on C whose endpoints are both in C. A graph is *strongly chordal* if every closed walk C of even length greater than four has an *odd chord*, which is a chord whose endpoints are an odd distance apart on C. Farber showed [8] that a graph is strongly chordal if and only if its vertices can be ordered so that the corresponding augmented adjacency matrix is Γ-free.

Our characterization of interval graphs as the reflexive, symmetric signed-interval digraphs is equivalent to the characterization stating that they are the reflexive min-orderable graphs. Although the relationship of co-TT graphs to min orderings has not previously been recognized, the equivalent orderings from [34] imply that co-TT graphs are strongly chordal [34].

Min orderings are a useful tool for graph homomorphism problems. A *homomorphism* of a digraph G to a digraph H is a mapping $f : V(G) \to V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in E(G)$. Digraph homomorphism problems are a special case of constraint satisfaction problems. A general tool for solving polynomial time solvable constraint

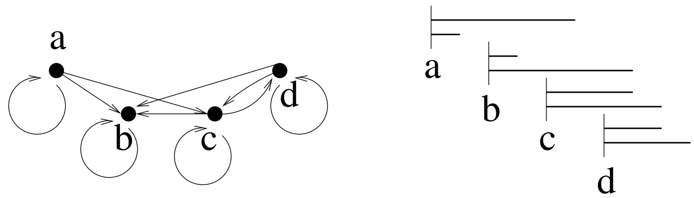

Figure 4: An adjusted interval digraph and a corresponding adjusted interval model. The source interval for each vertex is the upper one.

satisfaction problems are the so-called polymorphisms [4]. Without going into the technical details, we mention that min-orderings are equivalent to conservative semilattice polymorphisms [13]. In particular, if a digraph H has a min ordering, there is a simple polynomial-time algorithm to decide if a given input graph G admits a homomorphism to a fixed digraph H [19, 26]. In fact, the algorithm is well known in the AI community as the arc-consistency algorithm [4, 26]; it is easy to see that it also solves *list homomorphism* problems, where we seek a homomorphism of input G to fixed H taking each vertex of G to one of a 'list' of allowed images [10, 11, 12, 13]. In fact, many (but not all) homomorphism and list homomorphism problems that can be solved in polynomial time can be solved using arc-consistency with respect to a min ordering.

2 Previous work

Interval graphs are important in graph theory and in applications, and are distinguished by several elegant characterizations and efficient recognition algorithms [3, 10, 14, 16, 20, 31, 38]. One attempt to extend the concept to digraphs is given in [37], but many of the desirable structural properties are absent. More recently, the more restricted class of adjusted interval digraphs has been found to offer a nicer generalization of interval graphs [13]. Recall that digraph H is an adjusted interval digraph if there are two families of real intervals, the *source intervals* $\{[x_v, y_v]|v \in V(H)\}$ and the *sink intervals* and $\{[x_v, z_v]|v \in V(H)\}$ such that $uv \in E(H)$ if and only if the source interval for u intersects the sink interval for v. (See Figure 4.) This differs from the class in [37] in that the left endpoint, x_v, must be shared by the two intervals $[x_v, y_v]$ and $[x_v, z_v]$ assigned to v; they are "adjusted." An *adjusted interval model* of H is a set of source and sink intervals that represent H in this way.

An interval model of an interval graph G can be viewed as two mappings $\{v \rightarrow x_v|v \in V(H)\}$ and $\{v \rightarrow y_v|v \in V(H)\}$ such that $x_v \leq y_v$ for each $v \in V(H)$, and such that $uv \in E(H)$ if and only if $y_v \leq x_u$ and $y_u \leq x_v$; $[x_v, y_v]$ is the interval corresponding to v. The constraint $x_v \leq y_v$ comes from the need for $[x_v, y_v]$ to be an interval. The proposition that two intervals intersect is the same as the proposition ($x_v \leq y_u$ and $x_u \leq y_v$), since this means that neither interval lies entirely to the right of the other.

A generalization of interval models is obtained by dropping the constraint $x_v \leq y_v$ for each $v \in V(H)$ in this formulation, while retaining the constraint that uv is and edge if and only if $x_v \leq y_u$ and $x_u \leq y_v$. Recall that a graph H is a *threshold tolerance* *graph* [34] if each vertex v can be assigned a weight w_v and a tolerance t_v so that for all $a, b \in V(H)$, ab is an edge of H if and only if $w_a + w_b > \min(t_a, t_b)$. and the co-TT graphs are the complements of threshold tolerance graphs. A graph H is a co-TT graph, if there exist real numbers $x_v, y_v, v \in V(H)$, such that $ab \in E(H)$ if and only if $x_a \leq y_b$ and $x_b \leq y_a$ [18]. This differs from the definition of interval graphs in that it is no longer required that $x_v \leq y_v$, illustrating the motivation for dropping the constraint in this case. (See Figure 5.) That these are precisely the co-TT graphs is easily seen by letting $x_v = w_v$ and $y_v = t_v - w_v$. The two mappings $v \to x_v$ and $v \to y_v$, are called the *co-TT model* of H.

One view of a co-TT model is that there are now intervals whose 'beginning,' x_v, may come after their 'end,' y_v. In other words, we may have 'intervals' $[x_v, y_v]$ with $y_v < x_v$. We may view a co-TT model as consisting of intervals $[x_v, y_v], v \in V(H)$, some of which are oriented in the positive direction (have $x_v \leq y_v$) and others are oriented in the negative direction (have $x_v > y_v$). We speak of *positive* or *negative* intervals, and *positive* or *negative* vertices that correspond to them. (In the literature [15, 18, 23, 34], the direction is denoted by *colors* of the intervals: positive intervals, and vertices, are colored *blue*, and negative intervals, and vertices, are colored *red*.)

The definition of adjacency in a co-TT model implies that two positive vertices are adjacent if and only if they intersect; in particular, each positive vertex has a loop. Two negative vertices are never adjacent; in particular negative vertices have no loops. Finally, a positive vertex u corresponding to a positive interval $[a, b]$ and a negative vertex v corresponding to a negative interval $[c, d]$ are adjacent if and only if $[d, c]$ is contained in $[a, b]$ (i.e., $a \leq d \leq c \leq b$).

We emphasize that our definition of co-TT graphs differs from the standard definition [15, 18, 34]. In the standard definition, the condition $ab \in E(H) \iff x_a \leq y_b$ and $x_b \leq y_a$ is applied only for $a \neq b$, ignoring the issue of loops. We generalize the condition to the case where $a = b$, which can require that some of the vertices have loops. Thus, a graph under the standard interpretation is co-TT if and only if with a suitable addition of loops it is co-TT under our definition above. It is not necessary to know a co-TT model of the graph in order to convert a co-TT graph without loops into one satisfying our definition in linear time. The *closed neighborhood* of a vertex x, denoted $N[x]$, consists of x and its neighbors. Two vertices are *true twins* if they have identical closed neighborhoods. A vertex is *simplicial* if its closed neighborhood induces a complete subgraph. It was shown in [18] that if a graph H is co-TT (in the standard sense), then it has a co-TT model with negative intervals for all simplicial vertices without true twins and all other intervals positive. Thus, there is an easy translation between the co-TT graphs as defined here and the standard irreflexive co-TT graphs, namely, loops may be placed on all vertices other than simplicial vertices that have no true twins. A linear-time algorithm is given in [15] for performing this operation.

Note that the interval graphs are those co-TT graphs that have a co-TT model where all vertices are positive. In other words, they are the reflexive co-TT graphs.

Adjacency on a set of intervals can also be defined by interval containment. A graph is a *containment graph of intervals* [17] if there is a family of intervals $\{[x_v, y_v] | v \in V(H)\}$

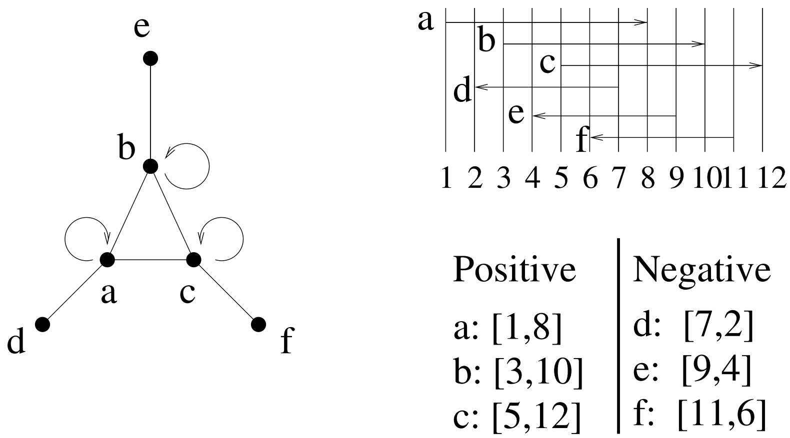

Figure 5: A co-TT graph and a corresponding co-TT model; ab is an edge since $1 \leq 10$ and $3 \leq 8$, ad is an edge since $1 \leq 2$ and $7 \leq 8$. However, bd is not an edge: although $7 \leq 10$, 3 is not less than or equal to 2. The example of this figure is one of the well-known minimal graphs that are not interval graphs, illustrating that the interval graphs are a proper subclass of the co-TT graphs.

on the real line such that $uv \in E(H)$ if and only if one of $[x_u, y_u]$ and $[x_v, y_v]$ contains the other. A graph is a containment graph of intervals if and only if it and its complement are both *transitively orientable*, thus if and only if it is a *permutation graph* [17].

A concept related to interval graphs for bipartite graphs is as follows. A bipartite graph H with parts A, B is an *interval bigraph* if there are intervals $\{[x_a, y_a], a \in A\}$, and $\{[x_b, y_b], b \in B\}$, such that for $a \in A$ and $b \in B$, $ab \in E(H)$ if and only if $[x_a, y_a] \cap [x_b, y_b] \neq \emptyset$. Such a set of intervals is known as an *interval bigraph* model of the graph. For this paper, a more relevant class is a bipartite version of this concept. A bipartite graph H with parts A, B is an *interval containment bigraph* [21, 29] if there are sets of intervals $\{I_a | a \in A\}$, and $\{J_b | b \in B\}$, such that $ab \in E(H)$ if and only if $J_b \subseteq I_a$. These graphs have been independently studied from the point of view of another geometric representation, defined as follows [36]. A bipartite graph H with parts A and B is called a *two-directional orthogonal ray bigraph* if there exists a set $\{U_a, a \in A\}$ of upwards vertical rays, and a set $\{R_b, b \in B\}$ of horizontal rays to the right such that $ab \in E(H)$ if and only if $U_a \cap R_b \neq \emptyset$. It is known that a bipartite graph is an interval containment bigraph if and only if its complement is a circular arc graph [11, 29] (and thus if and only if it is a two-directional orthogonal ray bigraph).

Matrices that can be permuted to avoid small submatrices have been of much interest [1, 30, 32]. This of course corresponds to characterizations of digraphs by forbidden ordered subgraphs [7, 24]. Our focus is on $\{\Sigma, \Lambda\}$-free matrices. A relationship between this and the previous work is described in Section 6.

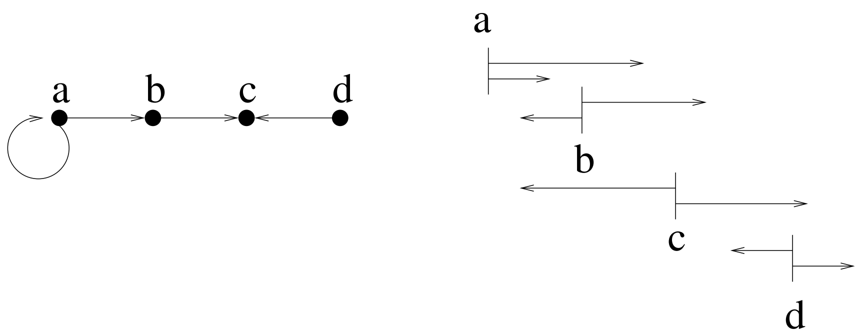

Figure 6: A signed-interval digraph and a corresponding signed-interval model. The source interval for each vertex is the upper one. There is a loop at a because its positive source interval intersects its positive sink interval. There is an edge from a to b because a's positive source interval contains b's negative sink interval, an edge from b to c because b's positive source interval intersects c's positive sink interval, and an edge from d to c because d's negative source interval is contained in c's positive sink interval.

3 Signed-interval digraphs and min orderings

We have now seen extensions of interval graphs in two different directions. First, taking two (adjusted) intervals instead of just one interval extends them to a class of digraphs. Second, by admitting negative intervals extends them to a broader class of (symmetric) graphs. Both these generalizations have proved very fruitful [10, 13, 15, 28, 18, 23, 34].

We now define a new class of digraphs that unifies these extensions, by assigning a source vertex and a sink vertex to each vertex, as in the adjusted interval model, and allowing these intervals to be either positive or negative, as in the co-TT model. In particular, a signed-interval model is obtained in by assigning, for each $v \in V(H)$, a *source interval* $[x_v, y_v]$ and a *sink interval* $[x_v, z_v]$, such that it is not required that $y_v, z_v \geq x_v$, and $uv \in E(H)$ if and only if $x_u \leq z_v$ and $x_v \leq y_u$. A graph is a *signed-interval digraph* if it can be modeled in this way. (See Figure 6.) The model can be viewed as three mappings from $V(H)$ to the real line, $v \to x_v, v \to y_v$, and $v \to z_v$. Since it is possible that $x_v > y_v$ and/or $x_v > z_v$, each of $[x_v, y_v]$ and $[x_v, z_v]$ can be negative or positive. Since the source interval and sink interval for v share the endpoint x_v, we retain the property that the intervals are adjusted.

Let H be a signed-interval digraph and consider a signed-interval model of H given by the ordered pairs (I_v, J_v) of intervals where $I_v = [x_v, y_v]$ and $J_v = [x_v, z_v]$. For $\alpha, \beta \in \{+, -\}$, we say a vertex v is of type (α, β) if I_v is an α-interval and J_v is a β-interval. The subdigraph of H induced by $(+,+)$-vertices is an adjusted interval digraph. The $(-,-)$-vertices of H form an independent set. The arcs between the $(+,-)$- and $(-,-)$-vertices form a 2DOR digraph. The arcs between the $(-,+)$- and $(-,-)$-vertices also form a 2DOR digraph. Similar properties hold for the other parts and their connections.

It has previously been recognized that interval graphs, adjusted interval digraphs, and two-directional orthogonal ray digraphs have min orderings when care is taken to specify

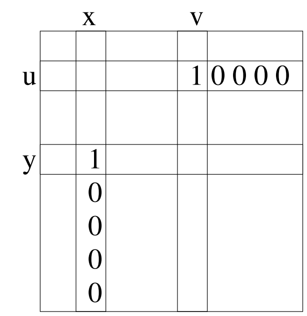

Figure 7: A matrix in a $\{\Sigma, \Lambda\}$-free ordering; v is the last out-neighbor $O(u)$ of u in the ordering and y is the last in-neighbor $I(x)$ of x in the ordering. The absence of an edge from u to x would violate the ordering property, since rows u, y and columns x, v would contain one of the matrices Σ, Λ, cf. Figure 2.

which vertices have loops and which do not [10, 13, 25, 36].

The main result of this section is the following.

Theorem 3.1. *A digraph admits a min ordering if and only if it is a signed-interval digraph.*

Before embarking on the proof we offer an alternate definition of a min ordering. Consider any linear ordering $<$ of $V(H)$. To this ordering, we prepend an initial element α, which is a place holder and not a vertex. Thus, $\alpha < x$ for each vertex x. Suppose the adjacency matrix is ordered according to $<$. For a vertex u, we denote by $O(u)$ the last vertex v (in the order $<$), such that v is an out-neighbor of u, or α if a has no out-neighbor. (See Figure 7.) Similarly, for each vertex x, we denote by $I(x)$ the last vertex y such that y is an in-neighbor of x, or α if a has no in-neighbor.

Proposition 3.2. *A linear ordering $<$ of $V(H)$ is a min ordering of a digraph H if and only if the following property holds:*

$$ux \in E(H) \textit{ if and only if } u \leq I(x) \textit{ and } x \leq O(u).$$

Proof. (See Figure 7.) Suppose first that $<$ is a min ordering of H with α prepended. If $ux \in E(H)$, then by the definition of $O(u), I(x)$ we have $u \leq I(x)$ and $x \leq O(u)$. On the other hand, let $u \leq I(x)$ and $x \leq O(u)$. Note that if $u = I(x)$ or $x = O(u)$ we have $ux \in E(H)$ also by definition. Therefore it remains to consider vertices u, x such that $u < y = I(x)$ and $x < v = O(u)$. Then $uv, yx \in E(H)$ and the min ordering property implies that $ux \in E(H)$. This proves the property.

Conversely, assume that $<$ is a linear ordering of $V(H)$ with α prepended and that the property holds for $<$. We claim it is a min ordering of H. Otherwise some $ab \in E(H), a'b' \in E(H), a < a', b' < b$ would have $ab' \notin E(H)$. This is a contradiction, since we have $a < a' \leq I(b')$ and $b' < b \leq O(a)$. $\square$

We proceed to prove the theorem.

Proof. Suppose $<$ is a min ordering of a digraph H with α prepended. We represent each vertex $v \in V(H)$ by the mappings $v \to v, v \to O(v), v \to I(v)$. In other words, v is represented by the two intervals $[v, O(v)]$ and $[v, I(v)]$. It follows from Proposition 3.2 that $ab \in E(H)$ if and only if $a \leq I(b)$ and $b \leq O(a)$. Thus, H is a signed-interval digraph.

Conversely, suppose we have the three mappings $v \to x_v, v \to y_v, v \to z_v$ from $V(H)$ to the real line, such that $ab \in E(H)$ if and only if $x_a \leq z_b$ and $x_b \leq y_a$. Without loss of generality we may assume the points $\{x_v | v \in V(H)\}$ are all distinct. Then we claim that the left to right ordering of the points x_v yields a min ordering $<$ of H, with a real point preceding these points corresponding to α. (Specifically, we define $a < b$ if and only if x_a precedes x_b.) Consider now $ab \in E(H), a'b' \in E(H)$, with $a < a', b' < b$. This means that $x_a < x_{a'} \leq z_{b'}$ and $x_{b'} < x_b \leq y_a$, whence we must have $ab' \in E(H)$. $\square$

In the construction of the proof, a vertex v is assigned a positive source interval if $O(v) > v$ and a negative one otherwise, and a positive sink interval if $I(v) > v$ and a negative one otherwise. By Proposition 3.2, if both of v's intervals are positive, v requires a loop, and it cannot have a loop if at least one of its intervals is negative.

4 An alternate geometric representation of signed-interval digraphs

Digraphs that admit a min ordering have another geometric representation. Let C be a circle with two distinguished points (the *poles*) N and S, and let H be a digraph. Let $I_v, v \in V(H)$ and $J_v, v \in V(H)$ be two families of arcs on C such that each I_v contains N but not S, and each J_v contains S but not N. We say that the families I_v and J_v are *consistent* if they have the same clockwise order of their clockwise ends, i.e., the clockwise end of I_a precedes in the clockwise order the clockwise end of I_b if and only if the clockwise end of J_a precedes in the clockwise order the clockwise end of J_b. Suppose two families I_v, J_v are consistent; we define an ordering $\prec$ on $V(H)$ where $a \prec b$ if and only if the clockwise end of I_a precedes in the clockwise order the clockwise end of I_b; we call $\prec$ the ordering *generated by* the consistent families I_v, J_v. Note that $\prec$ is a total order on $V(H)$.

A *bi-arc model* of a digraph H is a consistent pair of families of circular arcs, $I_v, J_v, v \in V(H)$, such that $ab \in E(H)$ if and only if I_a and J_b are disjoint. A digraph H is called a *bi-arc digraph* if it has a bi-arc model.

Theorem 4.1. *A digraph H admits a min ordering if and only if it is a bi-arc digraph.*

Proof. Suppose I_v, J_v form a bi-arc model of H. We claim that the ordering $\prec$ generated by I_v, J_v is a min ordering of H. Indeed, suppose $a \prec a'$ and $b' \prec b$ have $ab, a'b' \in E(H)$. Then $I_{a'}$ spans the area of the circle between N and the clockwise end of I_a, and J_b spans the area of the circle between S and the clockwise end of $J_{b'}$. (See Figure 1.) This implies that I_a and $J_{b'}$ are disjoint: indeed, the counterclockwise end of I_a is blocked from

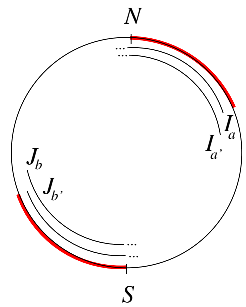

Figure 8: Illustration for the proof of Theorem 4.1

reaching $J_{b'}$ by J_b (since $ab \in E(H)$), and the counterclockwise end of $J_{b'}$ is blocked from reaching I_a by $I_{a'}$ (since $a'b' \in E(H)$). (The clockwise ends are fixed by the ordering $\prec$.)

Conversely, suppose $<$ is a min ordering of H. We construct families of arcs I_v and J_v, with $v \in V(H)$, as follows. The intervals I_v will contain N but not S, the intervals J_v will contain S but not N. The clockwise ends of I_v are arranged in clockwise order according to $<$, as are the clockwise ends of J_v. The counterclockwise ends will now be organized so that $I_v, J_v, v \in V(H)$, becomes a bi-arc model of H. For each vertex $v \in V(H)$, we define $O(v)$ and $I(v)$ as in the proof of Theorem 1. Then we assign the counterclockwise endpoint of I_v to be N if v has no out-neighbors, or else extend I_v counterclockwise as far as possible without intersecting $J_{O(v)}$, and assign the the counterclockwise endpoint of each J_v to be S if v has no in-neighbors, or else extend J_v counterclockwise as far as possible without intersecting $I_{I(v)}$. We claim this is a bi-arc model of H. Clearly, if $b > O(a)$, then I_a intersects J_b by the construction, and similarly for $a > I(b)$ we have J_b intersecting I_a. This leaves disjoint all pairs I_a, J_b such that $a \leq I(b)$ and $b \leq O(a)$; since $aO(a), I(b)b \in E(H)$, the definition of min ordering implies that $ab \in E(H)$, as required. □

Corollary 4.2. *The following statements are equivalent for a digraph H.*

- *H has a min ordering*
- *H is a signed-interval digraph*
- *H is a bi-arc digraph.* □

5 Bipartite graphs

Definition 5.1. *A bipartite graph G is a* signed-interval bigraph *if some uniform orientation H of G is a signed-interval digraph.*

We will show below that if some uniform orientation of a bipartite graph G is a signed-interval digraph, then so is every uniform orientation. If G is a signed-interval bigraph, then a signed-interval model of a uniform orientation H of G gives a representation of G: ab is an undirected edge of G if and only if one of ab and ba is an edge of H.

Note that a signed-interval bigraph is not necessarily a signed-interval digraph in the sense given previously. For signed interval bigraphs we must first assign a uniform orientation before considering whether the adjacency matrix has a $\{\Sigma, \Lambda\}$-free ordering. Once it is assigned, the rows and columns of the nonempty elements of the matrix are disjoint, which implies that so the rows and columns can be ordered independently.

The *bi-adjacency matrix* of a bipartite graph G with parts A, B has its i, j-th entry equal to 1 if and only if the i-th vertex in A is adjacent to the j-th vertex in B. Note that for this interpretation it is not required that the matrix be square.

Definition 5.2. *A 0-1 matrix has a* bipartite min ordering *if it has an independent permutation of rows and columns that is* $\{\Sigma, \Lambda\}$*-free.*

Lemma 5.3. *t A bipartite graph* $G = (A, B, E)$ *is a signed-interval bigraph if and only if its bi-adjacency matrix has a bipartite min ordering.*

Proof. Let C be a bi-adjacency matrix of a bipartite graph G, where A is its rows and B is its columns. Let H be a uniform orientation of G from A to B. An $n \times n$ adjacency matrix M for H can be obtained by moving the rows of A to the first $|A|$ rows of M, the columns of B in the last $|B|$ columns, and placing zeros elsewhere. Permuting the columns in A, does not change M, since they only contain zeros. Similarly, permuting the rows in B does not change M.

Suppose an independent permutation π_A of rows and π_B of columns of C produces a $\{\Sigma, \Lambda\}$-free matrix. The symmetric permutation π_A of both rows and columns of A and a symmetric permutation π_B of both rows and columns of B produces a $\{\Sigma, \Lambda\}$-free ordering of M.

Conversely, suppose H is a signed-interval digraph. There is a symmetric permutation of rows and columns of its adjacency matrix M that is $\{\Sigma, \Lambda\}$-free. Moving the rows in A to the first $|A|$ positions without changing their relative order and moving the columns of $|B|$ to the last $|B|$ positions without changing their relative order gives a $\{\Sigma, \Lambda\}$-free independent permutation of C in the first $|A|$ rows and last $|B|$ columns. □

Theorem 5.4. *The following statements are equivalent for a bipartite graph* H*.*

- H *is a signed-interval bigraph;*
- H *is a two-directional orthogonal ray bigraph;*
- *the complement of* H *is a circular arc graph*
- H *is an interval containment bigraph.*

Proof. The equivalence of the last three classes follows from a combination of results from [11, 25, 29, 36]. We complete the theorem by showing the equivalence, for bipartite graphs, of the signed-interval bigraphs and the two-directional orthogonal ray bigraphs. (Cf. also [25] where the third statement is shown equivalent to the existence of a min ordering.)

Suppose H has a signed-interval model given by the three mappings $v \to x_v, v \to y_v, v \to z_v$ such that $ab \in E(H)$ if and only if $x_a \leq z_b$ and $x_b \leq y_a$. We construct a two-directional ray model for H as follows. For each $a \in A$, we take an upwards vertical ray starting in the point P_a with x-coordinate equal to y_a and with y-coordinate equal to x_a. For each $b \in B$, we take a horizontal ray to the right, starting in the point Q_b with x-coordinate x_b and y-coordinate z_b. Now P_a intersects Q_b if and only if $x_b \leq y_a$ and $x_a \leq z_b$, i.e., if and only if $ab \in E(H)$ as required.

Now suppose that H has a two-directional model, i.e., upwards vertical rays $U_a, a \in A$, and horizontal rays to the right $R_b, b \in B$, such that $ab \in E(H)$ if and only if $U_a \cap R_b \neq \emptyset$. We will prove that H has a min ordering, whence it is a signed-interval digraph by Theorem 3.1. We will define the orders $<$ on A and on B as follows. Assume the starting point of the vertical ray U_a has the (x, y)-coordinates (u_a, v_a), and the starting point of the horizontal ray R_b has the (x, y)-coordinates (r_b, s_b), for $a \in A$, and $b \in B$. It is easy to see that we may assume, without loss of generality, that all $u_a, a \in A$, and $r_b, b \in B$ are distinct, and similarly for $v_a, a \in A$ and $s_b, b \in B$. We define $a < a'$ in A if and only if $v_a < v'_a$, and define $b < b'$ in B if and only if $r_b < r_{b'}$. We show that this is a min ordering of the bipartite digraph H. Otherwise, some $ab \in E(H), a'b' \in E(H), a < a', b' < b$ have $ab' \notin E(H)$. There are two possibilities for $ab' \notin E(H)$; either $u_a < r_{b'}$ or $u_a > r_{b'}, v_a > s_{b'}$. In the former case, $U_a \cap R_b = \emptyset$, in the latter case $U_{a'} \cap R_{b'} = \emptyset$, contradicting the assumptions. □

6 Special cases

We now explore what min orderings look like in the special cases we have discussed, namely reflexive graphs, reflexive digraphs, undirected graphs, and bipartite graphs. The results are all corollaries of Theorem 3.1 and Proposition 3.2.

Corollary 6.1. *A reflexive digraph H is a signed-interval digraph if and only if it is an adjusted interval digraph.* □

Next we focus on symmetric digraphs, i.e., graphs.

Corollary 6.2. *A reflexive graph H is a signed-interval digraph if and only if it is an interval graph. A graph H is a signed-interval digraph if and only if it is a co-TT graph.*

Proof. Consider an interval model or co-TT model of H, given by the mappings $v \to x_v, v \to y_v$, setting the third mapping $v \to z_v$ with each $z_v = y_v$, yields a signed-interval digraph model of H. Conversely, assume H is a graph, i.e., a symmetric digraph, that is a signed-interval digraph. Let $<$ be a min ordering of H; we again have $O(v) = I(v)$ for all vertices v. We claim that the mappings $v \to x_v = v, v \to y_v = O(v)$ define a co-TT model. Indeed, from Proposition 3.2 we have $ab \in E(H)$ if and only if $a \leq O(b) = y_b$ and

$b \leq O(a) = y_a$, as required. If, in addition, H is reflexive, then $O(v) = I(v) \geq v$, and $\{[v, O(v)], v \in V(H)\}$ is an interval model. □

Corollary 6.2 gives a novel way to understand the relationship between these classes.

By Corollary 6.2, a graph is an interval graph if and only if there is an ordering of vertices such that its augmented adjacency matrix is $\{\Sigma, \Lambda\}$-free. Also, by Corollary 6.2, a graph is a co-TT graph if and only if there is an ordering of vertices such that its adjacency matrix with some assignment of 0's and 1's to the elements of the diagonal is $\{\Sigma, \Lambda\}$-free. (One way to find such an assignment of 0's and 1's is the one given in [18].) Recall again that Farber proved [8] that a graph is strongly chordal if and only if there is an ordering of its vertices such that the corresponding augmented adjacency matrix is Γ-free. A comparison of all these statements offers a way to understand the relationship between interval graphs, co-TT graphs, and the broader class of strongly chordal graphs.

7 Algorithms and characterizations

Interval graphs are known to have elegant characterization theorems [14, 31], cf. [16, 38] and efficient recognition algorithms [3, 5, 20]. Thus, one might hope to be able to obtain similar results for their generalizations and digraph analogues. This is true for all the generalizations described in this paper, at least to some degree. In this section we summarize what is known.

The prototypical characterization of interval graphs is the theorem of Lekkerkerker and Boland [31]. In our language, it states that *a reflexive graph H is an interval graph if and only if it contains no asteroidal triple and no induced C_4 or C_5.* An *asteroidal triple* consists of three non-adjacent vertices such that any two are joined by a path not containing any neighbors of the third vertex. An equivalent characterization by the absence of a slightly less concise obstruction is given in [13]. *A reflexive graph H is an interval graph if and only if it contains no invertible pair.* An *invertible pair* is a pair of vertices u, v such that there exist two walks of equal length, P from u to v, and Q from v to u, where the i-th vertex of P is non-adjacent to the $(i+1)$-st vertex of Q (for each i), and also two walks of equal length R, S from v to u and u to v respectively, where the i-th vertex of R is non-adjacent to the $(i+1)$-st vertex of S (for each i). It is not difficult to see that an asteroidal triple is a special case of an invertible pair. A number of variants of the definition of an invertible pair have arisen [13, 15, 23, 25], and they have proved useful to give characterization theorems for various classes. It is proved in [13] that *a reflexive digraph is an adjusted interval digraph if and only if it contains no directed invertible pair.* A *directed version* of an invertible pair is defined in [13] in a manner similar to the above definition of an invertible pair. With yet another *labeled version* of an invertible pair, we have the following obstruction characterization of co-TT graphs: *a graph is a co-TT graph if and only if it contains no labeled invertible pair*, which follows from the characterization in [15] in terms of an interval ordering from [33]. For bipartite graphs, an analogous *bipartite version* of an invertible pair yields the following result. *A bipartite graph is a two-directional orthogonal ray bigraph if and only if it contains* *no bipartite invertible pair*, [25]. In fact, in [11] a stronger version is shown: there is a bipartite analogue of an asteroidal triple, called an *edge-asteroid*, and *a bipartite graph is a two-directional orthogonal ray bigraph if and only if it contains no induced* 6*-cycle and no edge-asteroid*. Bipartite graphs that contain no edge-asteroids are characterized in [23]. Finally, in [28], there is an obstruction characterization for signed-interval digraphs, which is a little more technical than just an invertible pair, [28].

There is a long history of efficient algorithms for the recognition of interval graphs, many of them linear time, starting from [3] and culminating in [5]. A polynomial time algorithm for the recognition of adjusted interval digraphs is given in [13]. It is not known how to obtain a linear time, or even near-linear time algorithm. An $O(n^2)$ algorithm for the recognition of two-directional orthogonal ray bigraphs follows from Theorem 5.4 and [33]. A more efficient algorithm in this case is also not known. On the other hand, an $O(n^2)$ algorithm for the recognition of co-TT graphs has been given in [15]. The obstruction characterization in [28] yields a polynomial-time algorithm for the recognition of signed-interval digraphs; in other words, to recognize whether a digraph has a min ordering.

References

[1] R.P. Anstee and M. Farber, Characterizations of totally balanced matrices, *J. Algorithms* 5 (1984) 215–230.

[2] V.L. Beresnev and A.I. Davydov, On matrices with connectedness properties, *Upravlyaemye Sistemy* 19 (1979) 3–13.

[3] K.S. Booth and G.S. Lueker, Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms, *J. Computer and System Sci.* 13 (1976) 335–379.

[4] A. Bulatov, P. Jeavons, and A. Krokhin, Classifying the complexity of constraints using finite algebras, *SIAM J. Computing* 34 (2005) 720–742.

[5] D.G. Corneil, S. Olariu, and L. Stewart, The LBFS structure and recognition of interval graphs, *SIAM J. Discrete Math.* 23 (2009) 1905–1953.

[6] V. Chvátal and P.L. Hammer, Set-packing and threshold graphs, *Univ. Waterloo Res. Report*, (1973) CORR 73-21.

[7] P. Damaschke, Forbidden ordered subgraphs, *Topics in Combinatorics and Graph Theory* (1990) 19–229.

[8] M. Farber, Characterizations of strongly chordal graphs, *Discrete Mathematics* 43 (1983) 173-189.

[9] M. Farber, Domination, independent domination, and duality in strongly chordal graphs, *Discrete Applied Mathematics* 7 (1984) 115–130.

[10] T. Feder and P. Hell, List homomorphisms to reflexive graphs, *J. Combinatorial Theory B* 72 (1998) 236–250.

[11] T. Feder, P. Hell, and J. Huang, List homomorphisms and circular arc graphs, *Combinatorica* 19 (1999) 487–505.

[12] T. Feder, P. Hell, and J. Huang, Bi-arc graphs and the complexity of list homomorphisms, *J. Graph Theory* 42 (2003) 61–80.

[13] T. Feder, P. Hell, J. Huang, and A. Rafiey. Interval graphs, adjusted interval digraphs, and reflexive list homomorphisms, *Discrete Applied Mathematics* 160 (2012) 697–707.

[14] D. R. Fulkerson and O. A. Gross. Incidence matrices and interval graphs, *Pacific J. Math.* 15 (1965) 835–855.

[15] P. Golovach, P. Heggerness. R.M. McConnell, V.F. dos Santos, J.P. Spinrad, and J.L. Szwarcfiter, On recognition of threshold tolerance graphs and their complements, *Discrete Applied Mathematics* 216 (2017) 171–180.

[16] M.C. Golumbic. *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York (1980).

[17] M.C. Golumbic and E.R. Scheinerman, Containment graphs, posets and related classes of graphs, in Combinatorial Mathematics, G. S Bloom et al., eds., Ann. NY Acad. Sci., 555 (1985) 192 - 204.

[18] M.C. Golumbic, N.L. Weingarten, and V. Limouzy, Co-TT graphs and a characterization of split co-TT graphs, *Discrete Applied Mathematics* 165 (2014) 168–174.

[19] W. Gutjahr, E. Welzl, and G.J. Woeginger, Polynomial graph-colourings, *Discrete Applied Mathematics* 35 (1992) 29–45.

[20] M. Habib, R.M. McConnell, C. Paul, and L. Viennot, Lex-BFS and partition refinement, with applications to transitive orientation, interval graph recognition and consecutive ones testing, *Theoretical Computer Science* 234 (2000) 59–84.

[21] P. Hell and J. Huang, Interval bigraphs and circular arc graphs, *J. Graph Theory* 46 (2004) 313–327.

[22] P. Hell, J. Huang, R.M. McConnell, and A. Rafiey, Interval-like graphs and digraphs, *MFCS 2018*, 69:1–69:13.

[23] P. Hell, J. Huang, R.M. McConnell, and J.C.-H. Lin, Comparability and cocomparability bigraphs, manuscript 2018.

[24] P. Hell, B. Mohar, and A. Rafiey, Orderings without forbidden patterns, In: Schulz A.S., Wagner D. (eds) Algorithms - ESA 2014. ESA 2014, *Lecture Notes in Computer Science*, vol 8737, Springer, Berlin, Heidelberg.

[25] P. Hell, M. Mastrolilli, M.M. Nevisi, and A. Rafiey. Approximation of minimum cost homomorphisms, *ESA 2012*, 587–598.

[26] P. Hell and J. Nešetřil *Graph Homomorphisms*, Wiley 2004.

[27] P. Hell and A. Rafiey, Monotone proper interval digraphs, *SIAM J. Discrete Math.* 26(4) (2012) 1576-1596.

[28] P. Hell, A. Rafiey, and A. Rafiey, Bi-arc digraphs and conservative polymorphisms, arXiv:1608.03368 v.4.

[29] J. Huang, Representation characterizations of chordal bipartite graphs, *J. Combinatorial Theory B* 96 (2006) 673–683.

[30] B. Klintz, R. Rudolf, and G.J. Woeginger, Permuting matrices to avoid forbidden submatrices, *Discrete Applied Mathematics* 60 (1995) 223–248.

[31] C.G. Lekkerkerker and J. C. Boland, Representation of a finite graph by a set of intervals on the real line, *Fundamenta Math.* 51 (1962) 45–64.

[32] A. Lubiw, Doubly lexical orderings of matrices, *SIAM J. Comput.* 16 (1987) 854–879.

[33] R.M. McConnell, Linear-time recognition of circular-arc graphs, *Algorithmica* 37 (2003) 93–147.

[34] C.L. Monma, B. Reed, and W. T. Trotter, Threshold tolerance graphs, *J. Graph Theory* 12 (1988) 343–362.

[35] R. Paige and R.E. Tarjan, Three partition refinement algorithms, *SIAM J. Comput.* 16 (1987) 973–989.

[36] A.M. Shresta, S. Tayu, and S. Ueno, On two-directional orthogonal ray graphs, *Proceedings of 2010 IEEE International Symposium on Circuits and Systems*, pp 1807–1810.

[37] M. Sen, S. Das, A.B. Roy, and D.B. West, Interval digraphs: an analogue of interval graphs, *J. Graph Theory* 13 (1989) 581–592.

[38] J.P. Spinrad, *Efficient Graph Representations*, Fields Institute Monographs, AMS 2003.