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Linear-Time Recognition of Circular-Arc Graphs[1]

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Abstract. A graph G is a *circular-arc graph* if it is the intersection graph of a set of arcs on a circle. That is, there is one arc for each vertex of G, and two vertices are adjacent in G if and only if the corresponding arcs intersect. We give a linear-time algorithm for recognizing this class of graphs. When G is a member of the class, the algorithm gives a certificate in the form of a set of arcs that realize it.

Key Words. Circular graph, Interval graph, Transitive orientation.

1. Introduction.

The *intersection graph* of a family of n sets is the graph where the vertices are the sets, and the edges are the pairs of sets that intersect. Every graph is the intersection graph of some family of sets [16]. A graph is an *interval graph* if there is a way to order the universe from which the sets are drawn so that each set is consecutive. Equivalently, a graph is an interval graph if it is the intersection graph of a finite set of intervals on a line. A graph is a *circular-arc graph* if it is the intersection graph of a finite set of arcs on a circle. (See Figure 1.) A *realizer* of an interval graph or circular-arc graph G is a set of intervals or circular arcs that represent G in this way.

An interval graph is a special case of circular-arc graphs; it is a circular-arc graph that can be represented with arcs that do not cover the entire circle. Some circular-arc graphs do not have such a representation, so the class of interval graphs is a proper subclass of the class of circular-arc graphs.

Interval graphs and circular-arc graphs arise in scheduling problems and other combinatorial problems. Before the structure of DNA was well understood, Benzer [1] was able to show that the set of intersections of a large number of fragments of genetic material in a virus were an interval graph. This provided strong evidence that genetic information was arranged inside a structure with a linear topology.

Being able to determine whether a graph is an interval graph or circular-arc graph constitutes *recognition* of these graph classes. However, having a representation of a graph with intervals or arcs can be helpful in solving combinatorial problems on the graph, such as isomorphism testing and finding maximum independent sets and cliques [4], [14]. Therefore, a stronger result than just recognizing the class is being able to produce the representation whenever a graph is a member of the class. In addition to its other uses, the representation constitutes a certificate that the graph is a member of the class.

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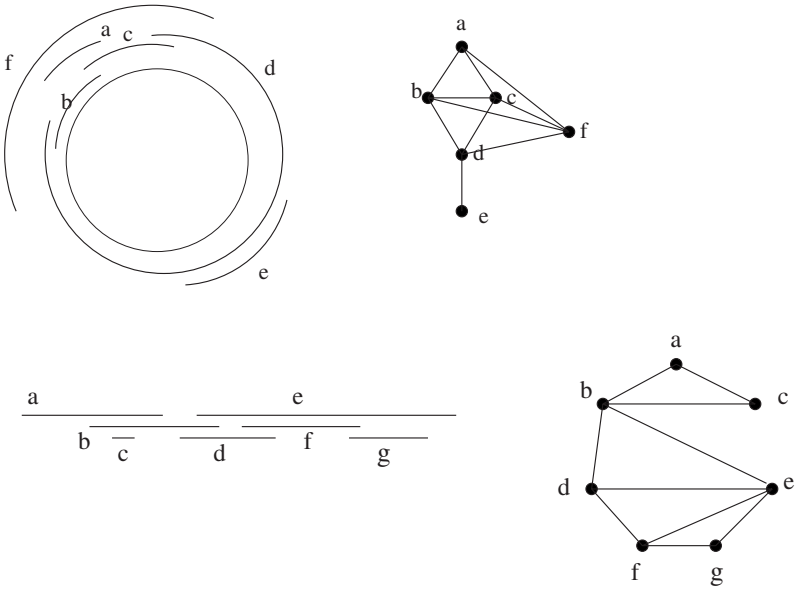


Fig. 1. A circular-arc graph is the intersection graph of a set of arcs on the circle, while an interval graph is the intersection graph of a set of intervals on the line.

Fulkerson and Gross [10] gave an $O(n^4)$ algorithm for solving this problem on interval graphs. Booth and Lueker later improved this to linear time [3]. Until now, a linear-time bound for circular-arc graphs has been elusive.

The reason that the problem is harder on circular-arc graphs than on interval graphs is that there are two ways to travel from one point to another on a circle, as opposed to just one on a line. When attempting to construct a realizer of a circular-arc graph, one must choose one of these when joining an adjacent pair of arcs, and the correct choice is not always obvious. The necessity of making these choices is absent in the interval-graph recognition problem. In an interval graph the maximal cliques correspond to regions of maximal overlap among the intervals, and there are therefore $O(n)$ maximal cliques. This plays an important role in Booth and Lueker's algorithm. This is not true of circular-arc graphs. For instance, three arcs can intersect pairwise around the circle, yet have no point in common. The number of maximal cliques in a circular-arc graph can be exponential in n [29].

It was initially conjectured by Booth [2] that recognition of circular-arc graphs was NP-complete. Tucker disproved this with an $O(n^3)$ algorithm [29]. Hsu improved this to $O(nm)$, where m is the number of edges [14], and Eschen and Spinrad further improved this to $O(n^2)$ [9].

In the current paper, which appeared in preliminary form in [17], we give an $O(n+m)$ time bound. Because this is linear in the size of the graph, further improvements to the time bound are not possible. Like the previous algorithms, this one produces a set of arcs that realize the graph whenever it is a circular-arc graph.

The algorithm is based on modular decomposition, transitive orientation of comparability graphs, and algorithms on permutation graphs and interval graphs. For overviews

of these topics, their applications, and their relationships to circular-arc graphs, see the books by Golumbic [12] and Roberts [25], and the survey article by Möhring [22].

2. Preliminaries. We may assume without loss of generality that no endpoints of arcs coincide, since in any realizer where they do, the endpoints can be moved by small amounts to make this true. To avoid cumbersome terms, we call the clockwise endpoint of an arc the *left* endpoint, and the counterclockwise endpoint the *right* endpoint. We can remember this relationship by thinking of the arc as "facing" the center of the circle. If x is a vertex of the graph, $l(x)$ and $r(x)$ denote the left and right endpoints of its arc. In contexts where the realizer is understood, we find it convenient to let x stand both for a vertex in G and for the arc $[l(x), r(x)]$ in the realizer.

If G is a circular-arc graph with realizer R, and X is a subset of the vertices, then $R|X$ is the *restriction of R to X*, namely, the result of removing from R those arcs corresponding to vertices not in X. Clearly, $R|X$ is a realizer of $G|X$.

We will operate on a realizer that is equivalent to a geometric one, namely, a circular ordering of $\{l(x) : x \in V\} \cup \{r(x) : x \in V\}$. The circular ordering represents the order of left and right endpoints as one travels counterclockwise around the circle. The circular ordering can be represented by an ordered list, where an incremental rightward movement is a movement from position i to position $i + 1 \pmod{2n}$. We call this the *circular-list* representation of the realizer. A circular-arc graph may have more than one realizer; when counting its realizers, we consider two realizers to be the same if one is a cyclic permutation of the other, and different otherwise. If G is an interval graph, we adopt the convention of circularly permuting the realizer so that an uncovered part of the circle occurs immediately following the last endpoint in the list. We call this the *ordered list* representation of an interval realizer. We consider two sets of intervals to be equivalent if their ordered-list representations are identical. Let the *mirror transpose* R^{T} of a realizer be the result of reversing the order of elements in the ordered list representation and reversing the roles of left and right endpoints.

Let $G = (V, E)$ be a graph. By $n(G)$ and $m(G)$ we denote the number of vertices and edges, respectively, of G. We use n and m for these if G is understood. Let $E' \subseteq G$ denote that $E' \subseteq E$, and let $e \in G$ denote that $e \in E$. $\bar{G}$ denotes the complement of G, whose vertices are V and whose edges are the nonedges of G. If G is a digraph, G^{T} denotes its transpose, which is obtained by reversing the directions of all of its directed edges. If v is a vertex in G, let $N(G, v)$ denote the neighbors of v in G. When G is understood, we may use $N(v)$ to denote this. $N[G, v]$ denotes the *closed neighborhood*, $N(G, v) \cup \{v\}$, and $N[v]$ denotes this when G is understood. Let $\bar{N}(G, v)$ denote the set $V - N[G, v]$ of *non-neighbors* of v, and let $\bar{N}(v)$ denote the same thing when G is understood.

If A is a matrix, we let A_{ij} denote the value in row i and column j of A. If $X \subseteq V$, $G|X$ denotes the *restriction of G to X*, namely, the graph $G' = (X, E \cap (X \times X))$. Similarly, if A is an $n \times n$ matrix and X is a subset of $\{1, 2, \ldots, n\}$, then $A|X$ is the $|X| \times |X|$ matrix of entries whose rows and columns are both in X. An $n \times n$ matrix can be represented with a directed edge-labeled graph on n vertices numbered 1 through n. The label of the directed edge from vertex i to vertex j is just the entry in row i and column j of the matrix. We may therefore refer to the *vertices* and *edges* of a matrix.

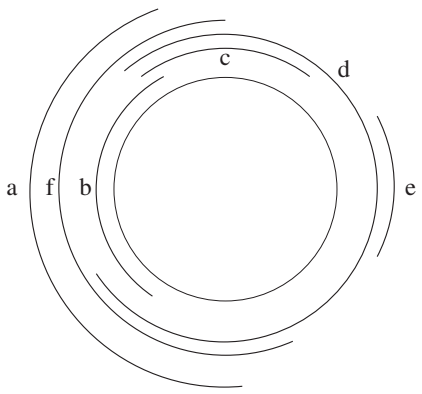
A *transitive orientation* of an undirected graph is an assignment of directions to its edges so that the resulting digraph is a transitive dag. A graph is a *comparability graph* if there exists a transitive orientation of it. Finding a transitive orientation of a comparability graph takes $O(n+m)$ time [18]. The complement of an interval graph is a comparability graph.

A *permutation graph* is a graph that is a comparability graph, and whose complement is also a comparability graph. The union of a transitive orientation of a permutation graph and a transitive orientation of its complement gives a linear order on the vertices. Reversing the orientations in the complement and again taking the union of the two gives another linear order. Two vertices are adjacent in the graph iff their relative order is the same in both of these orderings. This *permutation realizer* gives a way to represent a permutation graph with two orderings of the vertices.

3. Intersection Matrices. If G is a circular-arc graph and R is a realizer, we can classify each ordered pair (x, y) of vertices with $x \neq y$ according to the type of intersection of their arcs in R, as follows [29], [14], [9]:

- **Single overlap:** arc x contains a single endpoint of arc y.
- **Double overlap:** x and y jointly cover the circle and each contains both endpoints of the other.
- **Arc x is contained in arc y.**
- **Arc x contains arc y.**

Let $v_1, v_2, \ldots, v_n$ be a numbering of vertices of V. A *circular-arc matrix* of a realizer R is an $n \times n$ matrix T where T_{ij} is a label that tells the type of the relationship between arc v_i and arc v_j in the realizer. (See Figure 2.) R is a realizer of T if it realizes not just the graph implied by T but also the intersection types claimed by T. A circular-arc



	a	b	c	d	e	f
a		t	1	2	n	c
b	c		1	2	n	c
c	1	1		c	n	1
d	2	2	t		t	2
e	n	n	n	c		n
f	t	t	1	2	n	

Fig. 2. The intersection matrix of a circular-arc realizer gives the types of intersections between arcs. The types consist of two arcs overlapping at one endpoint, G_1 (1), two arcs overlapping at two endpoints, G_2 (2), two arcs not intersecting, G_n (n), one arc being contained in the other D_c (c) and one arc containing the other $(D_c)^{\mathrm{T}}$ (t). The pairs of each type give the graphs G_1, G_2, G_n, D_c, and $(D_c)^{\mathrm{T}}$, respectively. The matrix has a skew symmetry, where t in row i column j corresponds to c in row j column i.

matrix is an *interval matrix* if there is a realizer for it that does not cover the circle; in this case G is an interval graph, since a circular-arc realizer can be cut at an uncovered point on the circle and rolled out onto a line. An *intersection matrix* is a matrix whose entries give the alleged intersection types in some circular-arc realizer, but where it is not a requirement that the realizer exist. In general, we refer to a circular-arc matrix as an intersection matrix, except when we wish to emphasize that a realizer is required to exist.

From these types, we get a partition of the complete graph into the following undirected graphs on V:

- G_1: single overlaps.
- G_2: double overlaps.
- G_c: containments.
- $G_n = \bar{G}$: nonintersections.

We will use multiple subscripts to denote unions of these. For example $G = G_{12c}$, the union of edges of G_1, G_2, and G_c. An edge of G_c can be viewed as two directed edges with the following classifications:

- D_c: edges of the form $\{(x, y) : x \text{ is contained in } y\}$.
- $(D_c)^{\mathrm{T}}$: edges of the form $\{(x, y) : x \text{ contains } y\}$.

The *non-neighbors* of a vertex x are its neighbors in G_n, its *overlap neighbors* are its neighbors in G_1, its *double-overlap neighbors* are its neighbors in G_2, and its *containment neighbors* are its neighbors in G_c. $G(T)$ denotes the graph that gives the pairs that intersect, hence $G(T) = G_{12c}$. Its *neighbors* are the neighbors in $G(T)$.

A departure of our approach from previous ones is the use of a simple operation on the intersection matrix, which we call a *flip*. Recall that we consider a realizer to be a circular list giving the order of left and right endpoints of the vertices' arcs around the circle. Let a *geometric flip* on a vertex be an exchange of the positions of $l(x)$ and $r(x)$ in this list. This is equivalent to making x's arc travel between the same endpoints as before, but around the opposite part of the circle as before (Figure 3). This changes the types of relationships involving x's arc as follows:

- $(x, y) \in G_2$ becomes $(x, y) \in D_c$ and $(x, y) \in D_c$ becomes $(x, y) \in G_2$.
- $(x, y) \in G_n$ becomes $(x, y) \in (D_c)^{\mathrm{T}}$ and $(x, y) \in (D_c)^{\mathrm{T}}$ becomes $(x, y) \in G_n$.
- $(x, y) \in G_1$ remains unchanged.

If T is an intersection matrix, we can compute the intersection matrix T' for the realizer that results from a flip on x without any knowledge about the realizer, other than what is given directly by T. It requires only a simple relabeling of some of the entries of x's row and column in T. We call this relabeling of T an *algebraic flip* on x.

One matrix being obtainable from another by a sequence of algebraic flips is an equivalence relation on intersection matrices. We call this relation *flip-equivalence*. Every flip-equivalence class on circular-arc matrices contains an interval matrix: in a circular-arc realizer of a matrix T, if one picks a point on the circle that does not coincide with an endpoint of an arc and flips all arcs containing the point, the resulting set of arcs will fail to cover the circle at that point, and realize the interval matrix T' obtained from T by the equivalent algebraic flips.

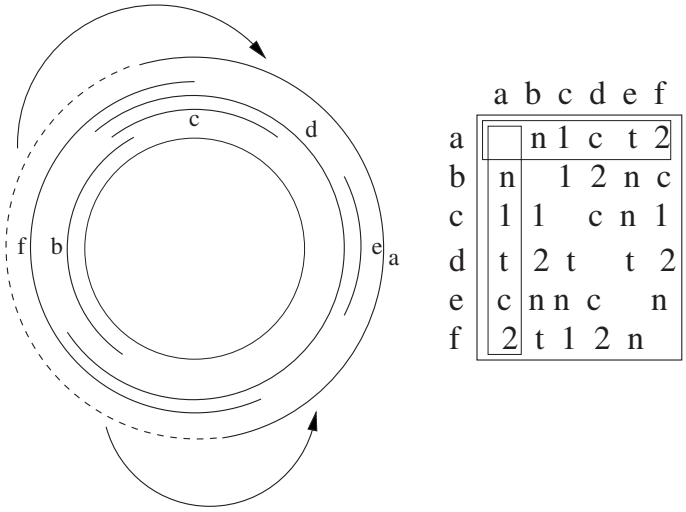


Fig. 3. A *geometric flip* consists of rerouting an arc's path between its endpoints; in this case arc a is flipped. (Compare with Figure 2.) An *algebraic flip* is the corresponding operation on the intersection matrix. An algebraic flip swaps the roles of n and t, and 2 and c in a row, and swaps the roles of n and c, and 2 and t in a column.

3.1. *Relationship to Previous Work*. It is tempting to try to transpose parts of our problem to another class of graphs. Eschen and Spinrad [9] compute the intersection constraints by transposing this problem into the domain of chordal bipartite graphs. Hsu [14] approaches the recognition problem by transposing much of it to the domain of circle graphs. The significance of flip operations is that they allow us to transpose much of the problem to the domain of comparability graphs, where we make use of existing analytical and algorithmic tools for that graph class.

The insights of Tucker [29], Hsu [14], and Eschen and Spinrad [9] are critical to obtaining our result. Tucker recognized the importance of the intersection types. Eschen and Spinrad developed a set of algorithmic techniques for computing intersection types efficiently in circular-arc graphs, of which we make extensive use in Section 7. Hsu showed that Tucker's intersection types can all be satisfied simultaneously.

There is a relationship between flipping of arcs and an operation applied by Tucker [28] to a subset of circular-arc graphs, and applied by Booth and Lueker [3] to circular-ones testing of matrices. A matrix has the consecutive-ones property if there is a way to arrange the columns so that the ones are consecutive in every row. It has the circular-ones property if there is a way to arrange the columns so that *either* the ones or the zeros are consecutive in each row. The operation consists of complementing those rows in a circular-ones matrix that have a one in a particular column, in order to obtain one that has the consecutive-ones property. This resembles the operation of picking a point on the circle in a circular-arc model and flipping arcs that contain it in order to obtain an interval model. However, though the circular-ones variant of the trick can be used to recognize certain subclasses of circular-arc graphs, such as *Helly circular-arc graphs*, it is not suitable for the general problem.

4. Summary of the Circular-Arc Graph Recognition Algorithm. In Section 5 we show that it is possible to check in time linear in the size of a graph G plus the size of a circular-arc realizer R whether G is the graph realized by R. Our approach to recognizing circular-arc graphs is then to give an algorithm that has, as a precondition, that its input be a circular-arc graph G, and that produces a circular-arc realizer of G in linear time whenever the precondition is met. As long as this algorithm halts when its precondition is not met, it suffices to recognize circular-arc graphs: we may check whether its output, if it produces any, is a circular-arc realizer of G. Since no circular-arc realizer can realize G if G is not a circular-arc graph, the test succeeds iff G is a circular-arc graph.

We now consider the time bound. If G is a circular-arc graph, our recognition algorithm runs in linear time. That is, there exists some constant c such that the algorithm halts before $c(n+m)$ operations have been executed. Therefore, there exists an algorithm that runs our algorithm and halts, rejecting G, if the number of operations executed exceeds $c(n+m)$. This proves that circular-arc graphs can be recognized in linear time. This result is the primary goal of the paper.

It is worth noting that a programmer who wishes to implement the algorithm would wish to have conventional halting conditions that do not require counting operations or a careful analysis of the constant hidden in our big-O bound. However, for most steps, the analysis of the time bound does not depend on whether the input graph is a circular-arc graph. Other steps have a precondition that be met if the input is a circular-arc graph, so that the input graph may be rejected early if the precondition is not met. Henceforth, we assume throughout the analysis that G is a circular-arc graph, except where we address this issue.

Algorithm 4.1 summarizes the approach to producing a circular-arc realizer.

ALGORITHM 4.1. Constructing a circular-arc realizer.

1. Find an intersection matrix T that can be shown to be realized by some realizer of G.
2. Perform a set of algebraic flips on T to obtain an interval matrix T'.
3. Find an interval realizer R' of T'.
4. Invert the flips used to obtain T' from T, but apply them as geometric flips to R', to obtain a circular-arc realizer R of T.

The fourth subproblem is trivial. We summarize the approach to the other three now, and give the full details in later sections.

4.1. *Step* 1: *Finding an Intersection Matrix that Gives the Intersection Types for Some Realizer of* G. A *universal vertex* is a vertex x with $N[x] = V$. A *module* of an undirected graph is a set X of vertices such that for every vertex $y \notin X$, either y is a neighbor of every member of X or it is a neighbor of none of them. A *clique module* is a module of G that is a (not necessarily maximal) clique in G.

We use a reduction that allows us to assume that G has no universal vertex or clique module. The following then gives the basic recipe for producing the desired intersection matrix.

DEFINITION 4.2. Let $T(G)$ denote the $n \times n$ matrix T of labels that is defined as follows:

1. If x and y are nonadjacent, then $T_{xy} = \texttt{n}$.
2. Else if $N[x] \subset N[y]$, then $T_{xy} = \texttt{c}$.
3. Else if $N[y] \subset N[x]$, then $T_{xy} = \texttt{t}$.
4. Else if $N[x] \cup N[y] = V$ and for each $z \in N[x] - N[y]$, $N[z] \subset N[x]$ and for each $z' \in N[y] - N[x]$, $N[z'] \subset N[y]$, then $T_{xy} = \texttt{2}$.
5. Else $T_{xy} = \texttt{1}$.

Hsu has shown that if G is a circular-arc graph without universal vertices or clique modules, any realizer of the intersection matrix $T(G)$ given by Definition 4.2 is a realizer of G [14]. It is not hard to see why this might be the case. If $N[x] \subset N[y]$, then in a realizer where arc x is not contained in arc y, one endpoint of x may be shortened to make it be contained in y without causing x to lose any neighbors. If x and y have a double overlap, then they jointly cover the circle, hence $N[x] \cup N[y] = V$. However, even when $N[x] \cup N[y] = V$, x and y cannot have a double overlap if there exist $z \in N[x] - N[y]$ and $z' \in N[y] - N[x]$ that are adjacent to each other. The x and y must fail to contain the intersection of arcs z and z', hence they cannot jointly cover the whole circle. The subtle point of Hsu's result is showing that there is a single realizer of G that simultaneously obeys all of these constraints.

This reduces the step to the problem of evaluating the boolean expressions of parts 2, 3, and 4 of Definition 4.2 at each edge xy of G.

We use a sparse representation of the matrix where we label the edges of G with their intersection types, which must fall into Cases 2–5. This saves us from having to spend time on entries of the matrix corresponding to Case 1. When we flip a vertex a, we must therefore add, delete, and relabel edges from this graph. This requires keeping a pointer from each adjacency-list entry (a, b) to its twin (b, a), and implementing the adjacency lists with doubly-linked lists.

Eschen and Spinrad have given $O(n^2)$ bounds for evaluating $N[x] \subset N[y]$, $N[y] \subset N[x]$, and $N[x] \cup N[y] = V$ at all pairs $\{x, y\}$ of vertices. We show how to apply their technique in $O(n + m)$ time if one needs to evaluate it only at the m adjacent pairs of G.

This allows us to produce $T(G)$, except that it does not allow us to distinguish between Cases 4 and 5 if $N[x] \cup N[y] = V$. That is, it does not allow us to tell in this case whether two adjacent arcs have a single overlap or a double overlap. The key insight for solving this problem is that if x and y have a double overlap, then flipping y causes y's interval to be contained in x's, hence it causes $N[y]$ to become a subset of $N[x]$. Flipping y reduces the problem to the foregoing one of testing whether a neighborhood containment applies between a pair of adjacent vertices, but on a modified graph where x is flipped.

The details of this step are given in Section 7.

4.2. *Step* 2: *Finding a Set of Flips to Turn* T *into an Interval Matrix.* The goal of this step is to identify the set of vertices whose arcs contain a point on the circle in a realizer of the intersection matrix. Flipping them vacates this part of the circle, giving an interval matrix.

What makes the step difficult is that, unlike in the case of an interval graph, the vertices of a clique do not have to have a common intersection point. For instance, the arcs $[0, \pi]$,

$[2\pi/3, 5\pi/3]$, and $[3\pi/2, \pi/2]$ form a clique but have no common intersection point on the circle. Therefore, it does not suffice to identify a maximal clique.

However, suppose that a vertex v_0 has no incident edges in G_d and G_c. Then every neighbor of the vertex in G is a neighbor in G_1. In a realizer, every neighbor contains one and only one endpoint of this vertex's arc. Those that contain one endpoint have a common intersection and those that contain the other have a common intersection. We then need only to identify the neighbors that contain one of the endpoints.

Such a v_0 might not exist initially, but it is easy to obtain one with some initial flips in order to isolate it in G_d and G_c. The one that we create has degree $O(m/n)$. This allows us to spend $O(n)$ time on each neighbor of the vertex without violating the linear time bound. This is adequate to be able to perform an arbitrary set of flips on neighbors of v_0.

To identify which neighbors of v_0 cover one endpoint, we use a combination of constraints imposed by their relationships to each other and by their relationships to the non-neighbors. For instance, two neighbors that have a G_d or G_n relationship to each other must cover opposite endpoints. Let W be the non-neighbors. Suppose x and y are neighbors of v_0 that have properly overlapping sets of neighbors in W. That is, $N[x] \cap W$ and $N[y] \cap W$ intersect but neither of these sets contains the other. Then x and y must obviously cover opposite endpoints of v_0 in a circular-arc realizer.

In practice, our approach is to begin to partition neighbors of v_0 according which endpoint of v_0 they cover. Partway through this process, we observe that the set of arcs that contain a third point on the circle is easy to identify. This allows us to quit early, since this set of arcs is also a solution to the problem.

The details of this step are given in Section 8.

4.3. *Step* 3: *Finding an Interval Realizer of an Interval Matrix.* It is easily seen that if T is an interval matrix, G_n is a comparability graph: the order of left endpoints in an interval realizer is a linear extension of a transitive orientation. (See Figure 4.) G_{1n} is a comparability graph for the same reason. In an interval matrix, G_2 is empty, since a double overlap of two arcs covers the entire circle. Therefore, G_c is the complement of G_{1n}. Since it also has a transitive orientation, namely D_c, it follows that G_c and G_{1n} are complementary comparability graphs, hence permutation graphs.

In [18] an algorithm is given for interval-graph recognition that uses a linear extension of a transitive orientation of $\bar{G} = G_n$. Thus, we can get an interval realizer of G in this way. Unfortunately, because of the added constraints in the types of intersections imposed by T, this might fail to be a realizer of T.

The permutation-graph recognition algorithm of [18] uses the transitive orientation algorithm to compute linear extensions of transitive orientations D and D' of a permutation graph H and its complement $\bar{H}$. It then finds the linear orders $D \cup D'$ and $D \cup (D')^{\mathrm{T}}$ in linear time, which gives the permutation realizer. Running this algorithm gives a linear extension of D_c and a linear extension of a transitive orientation G_{1n}, and therefore gives a permutation realizer of G_{1n}. It is easy to see that these two permutations are the order of appearance of left endpoints and the order of appearance of right endpoints in an interval realizer of G. However once again, this may fail to be an interval realizer of T.

The key observation we use is that every realizer of T gives a single orientation of G_{1n} that is *simultaneously* transitive in G_{1n} and G_n. Finding a transitive orientation of G_{1n} that is simultaneously transitive in G_n further constrains the possible solutions, and

allows us to get the orders of appearance of left and of right endpoints of a realizer of T. Interleaving these orders to produce the full realizer is a trivial operation. We show how the transitive orientation algorithm of [18] can be modified to produce the simultaneous transitive orientations of G_{1n} and G_n.

The details of this step are given in Section 6.

5. Verifying a Proposed Realizer. The correctness of our analysis depends on the claim, made earlier, that it is possible to verify in time linear in the size of G plus the size of a realizer R that R realizes G. We assume that R is presented as a list of arc endpoints and the vertices to which they belong, in the order in which they occur counterclockwise about the circle. When G is a circular-arc graph, our algorithm produces the realizer in this format.

ALGORITHM 5.1. Checking whether a circular-arc realizer realizes G.

Pick a starting point on the circle and make a doubly linked list L of all arcs that contain that point, in $O(n)$ time. Then travel counterclockwise, updating L whenever an endpoint of some arc x is encountered, so that it reflects the set of arcs that contain the current point. This requires adding x to L in $O(1)$ time if the encountered endpoint is a left endpoint, and removing it in $O(1)$ time (L is doubly linked) if the encountered endpoint is a right endpoint. Regardless of whether the endpoint is a left or right endpoint, generate the following edge records: $\{(y, x) : y \in L\}$ and $\{(x, y) : y \in L\}$. Halt when either the starting point on the circle has been reached again, or the total number of generated edge records exceeds $8m$. Reject the realizer if the number of generated edge records exceeds $8m$. Otherwise, radix sort the generated edge records and throw out duplicates. Then concatenate the adjacency lists of G and radix sort them. Verify that these two operations produce identical lists.

For the correctness and time bound, note that if x and y are adjacent, each contains at most two endpoints of the other. Therefore, if the realizer is valid, at most four copies of (x, y) will be generated, and at most four copies of (y, x) will be generated. The realizer can be rejected if the number of generated edge records exceeds $8m$. The radix sorts takes $O(n + m)$ time, and comparing the lists establishes whether G is identical to the graph represented by the realizer. The algorithm takes $O(n)$ time for all updates of L, and $O(1)$ time for each generated edge record, for a total of $O(n + m)$.

6. Details of Step 3: Finding an Interval Realizer of an Interval Matrix. Even though this section deals with the third step of Algorithm 4.1, we give it before the sections on the first two steps. It is of greatest interest to most readers, and develops a result that is used in a less obvious way in Step 2. (However, Section 6.4.5 is of interest primarily to specialists, and can be skipped without loss of continuity.)

Recall that an interval matrix is the intersection matrix of an interval realizer. In this section we deal with the problem of finding an interval realizer of an interval matrix. Since intervals on a line cannot realize a double-overlap relation, we may assume that the matrix is devoid of entries labeled 2 for the G_2 relation.

6.1. *Basic Tools.* An undirected graph is a special case of a directed graph where each undirected edge consists of two oppositely directed edges. In this paper we consider an $n \times n$ matrix A to be synonymous with a complete graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$, where each directed edge (v_i, v_j) is labeled with A_{ij}.

A *module of a matrix* corresponds to a set X of vertices such that for every vertex $y \notin X$, all directed edges in $X \times \{y\}$ have the same label, and all directed edges in $\{y\} \times X$ have the same label. In this case, y fails to *distinguish* members of X. A module of a graph or digraph $G = (V, E)$ is a module in its boolean adjacency matrix. That is, it is a set of vertices such that for each $y \notin X$, every element of $X \times \{y\}$ is an edge or none is, and every element of $\{y\} \times X$ is an edge or none is. Modules of matrices are studied in [6] and [7]. They were previously known in the special case of graphs, and first studied in [11].

The set V and its singleton subsets are *trivial modules*. A graph or matrix with only trivial modules is called *prime*.

A *modular partition* of V in a matrix is a partition of V where every partition class is a module. If X and Y are disjoint modules, then every element of $X \times Y$ has the same label. The *modular quotient* induced by the parts is the matrix obtained by making one vertex for each part, and letting the label of each ordered pair (X, Y) of parts be the labels of the edges from X to Y. One way to represent a modular quotient is with the submatrix induced by any set P consisting of one vertex from each part. Similarly, if R is a circular-arc realizer, then the quotient is realized with the smaller realizer $R|P$.

Two sets A and B *overlap* if $A \cap B$, $A - B$, and $B - A$ are all nonempty. A module is *strong* if it overlaps no other module. The transitive reduction of the containment relation on strong modules of a graph $G = (V, E)$ is a tree, and is called the *modular decomposition*. The root of the modular decomposition is V and the leaves are its singleton subsets. If G is undirected, at most one of G and its complement, $\bar{G}$, can be disconnected. If one of them is disconnected, the children of the root are the connected components, and V is a *degenerate node*. The remaining nodes are *prime nodes*. The family $\mathcal{F}$ of modules of G consists of those sets that are nodes of the tree, and those sets that are a union of siblings that have a degenerate parent. Let $MD(G)$ denote the modular decomposition of an undirected graph G.

The modular decomposition of a symmetric matrix is defined in the same way, except that the root is degenerate if the complement of the graph consisting of those edges with one label is disconnected. Let $MD(T)$ denote its modular decomposition of a symmetric matrix T.

An undirected graph can be viewed as a special case of a directed graph, where each undirected edge ab represents two directed edges, (a, b) and (b, a). If (a, b) and (c, d) are two directed edges, we say that $(a, b)\Gamma(c, d)$ iff $a = c$ and b and d are nonadjacent, or $b = d$ and a and c are nonadjacent. If G is a comparability graph, then $(a, b)\Gamma(c, d)$ implies that in any transitive orientation of G, either both of (a, b) and (c, d) appear as directed edges, or neither does. To understand why, suppose that $(a, b)\Gamma(c, d)$, but (a, b) and (d, c) are directed edges in an orientation of G. Suppose without loss of generality that $a = c$. Then (d, c) and (c, b) require a transitive directed edge (d, b), but d and b are nonadjacent, so this is impossible.

The transitive closure of the Γ relation is an equivalence relation on directed edges, and the equivalence classes are groups of directed edges, called *implication classes*, that

must either all appear in or all be absent from any transitive orientation of G. Therefore, if G is a comparability graph, (a, b) and its transpose (b, a) cannot be in the same implication class. The implication class containing (a, b) consists of the transposes of the directed edges in the implication class containing (b, a). The union of an implication class and its transpose is a *color class*. The color classes are a partition of the undirected edges of G.

There is a type of dual relationship between the modules of G and its color classes.

LEMMA 6.1 (see, for example, [22]). *If X and Y are disjoint nodes of the modular decomposition of G, then $X \times Y$ is a subset of a single implication class.*

The vertices spanned by a color class are always a module of G. Moreover, if M is a module, an edge of G that is in $G|M$ is always in a different color class from an edge that is not in $G|M$. Because of this, given a transitive orientation of G and its modular decomposition, one may obtain a new transitive orientation by reversing the directions of all edges inside a node of the modular decomposition tree. In a degenerate node whose children are nonadjacent in $\bar{G}$, a transitive orientation of G determines a linear order on the children. Reversing the orientations of all edges between a pair of children that are adjacent in this linear order gives a new transitive orientation. All transitive orientations of G are obtainable from a single one by applying these two reversal operations in different places in the modular decomposition tree.

It is not the case that every interval realizer of an interval graph G realizes the intersection matrix given by a particular realizer R of G. This is because its partition of edges of G into D_c, G_1, and G_2 relations given by a different realizer R' of G may not be faithfully reflected by the types of intersections in R's matrix. Therefore, an intersection matrix places additional constraints on possible realizers that G does not place, and not all realizers of G satisfy the requirements of Step 3.

6.2. *Interval Orientations.* Let T be an interval matrix and let R be an interval realizer of T. If $ab \in G_{1n}$, we say that *a precedes b* in R if $l(a) < l(b)$, even if they overlap at one endpoint. (Because $ab \in G_{1n}$, this happens iff $r(a) < r(b)$.) Let the *interval orientation* of G_{1n} given by a realizer R be the orientation of its edges such that for each edge $ab \in G_{1n}$, (a, b) appears as a directed edge in the orientation iff a precedes b in R. (See Figure 4.) We call matrix T' an interval orientation of T if it is obtained by replacing elements of T labeled G_1 with D_1 or $(D_1)^{\mathrm{T}}$, elements labeled G_n with D_n or $(D_n)^{\mathrm{T}}$, so that the elements labeled D_1 and D_n give an interval orientation of G_{1n}, and the elements labeled $(D_1)^{\mathrm{T}}$ and $(D_n)^{\mathrm{T}}$ are its transpose.

An interval orientation of G_{1n} gives a simultaneous transitive orientation of both G_n and of G_{1n}. To see why, note that if $ab, bc \in G_n$, then a and b are nonadjacent and b and c are nonadjacent. If a comes before b in a realizer and b comes before c, then a and c are nonadjacent, and a comes before c, providing a transitive edge (a, c) for the orientations (a, b) and (b, c) given by the linear extension. A similar argument applies to G_{1n}.

Therefore, G_n and G_{1n} are comparability graphs. In an interval matrix, G_2 is empty, so G_c is the complement of G_{1n}. G_c is also a comparability graph, since it has D_c as a transitive orientation. Since G_c and G_{1n} are complementary comparability graphs, they are permutation graphs.

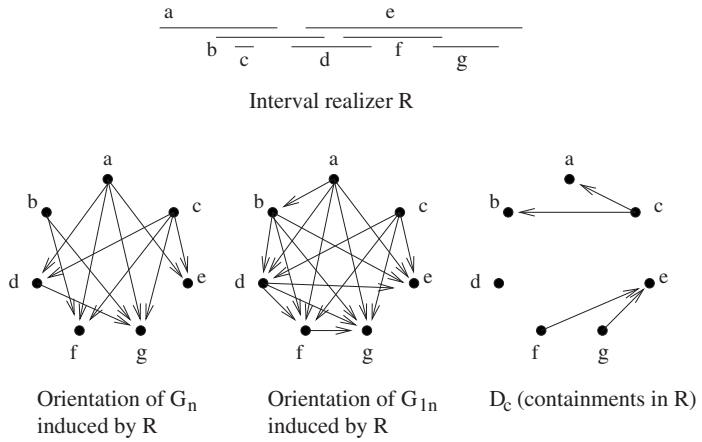


Fig. 4. An interval realizer of an intersection matrix yields an *interval orientation* of G_{1n}: edges are oriented from earlier to later intervals. An interval orientation is transitive, and its restriction to G_n is also transitive. The union of D_c and an interval orientation is a linear order, and gives the order of left endpoints in the realizer. Similarly, the union of $(D_c)^{\mathrm{T}}$ and the interval orientation gives the order of right endpoints.

The union of D_{1n} and D_c is a linear order, and gives the relative order of left endpoints in the realizer, and, similarly, the union of D_{1n} and $(D_c)^{\mathrm{T}}$ gives the relative order of right endpoints. This is essentially the permutation-graph recognition algorithm of [18], but places the additional restriction that the transitive orientation of G_{1n} be an interval orientation. It is easy to interleave the two resulting linear orders to recover the realizer.

This reduces the problem of finding a realizer of an interval matrix to one of finding an interval orientation. This has the advantage that it allows us to apply tools and concepts that were developed for the transitive orientation problem.

A module of a graph can be viewed as the result of a substitution operation that is depicted in Figure 5. As we have mentioned in Section 2, the edges that are internal to a module in a comparability graph may be oriented independently of those that are not when one wishes to compute a transitive orientation. (One must be careful not to create directed cycles among edges internal to different modules, however.) The modules completely specify the sets of edges that can be oriented independently of others [11], [22]. There is a structure that has an analogous role in the problem of finding interval orientations of G_{1n}, and which we call Δ modules.

To get an intuitive notion of a Δ module, we define an operation on interval realizers that is analogous to the substitution operation on graphs, and which is depicted in Figure 6. R_3 is the result of substituting R_2 for x in R_1, by replacing the left endpoint of x with the left endpoints of intervals in R_2 and the right endpoint of x with the right endpoints of intervals in R_2. A requirement when the two endpoints of x are not consecutive in R_1 is that all left endpoints of R_2 all precede all right endpoints in R_2, hence the interval graph represented by R_2 is a clique. Note that R_2 becomes a module in the resulting interval matrix. A second substitution operation that does not require R_2 to represent a clique is illustrated by R_4; a requirement of this substitution operation is that the two endpoints of the replaced interval in R_1 are consecutive.

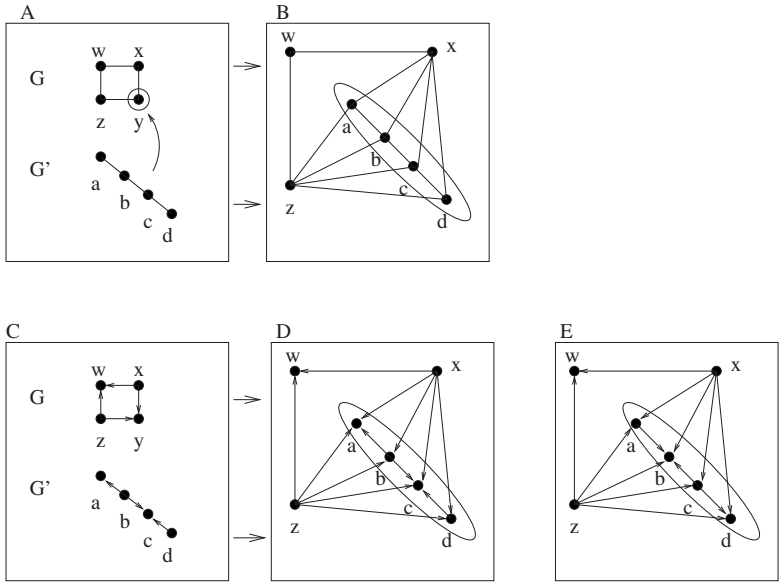


Fig. 5. A module of a graph can be viewed as the result of a substitution operation on two graphs G and G', where a vertex y of G is replaced with G', and each vertex in G' inherits the adjacencies of y in G (A, B). If G and G' are transitively oriented, the result of the substitution is also transitive (C, D). The transpose of a transitive orientation of G' is also transitive, so a new transitive orientation of the composite graph may be obtained by reversing the orientations of edges internal to a module (E).

A Δ *module* is a module of an intersection matrix that can result from one of these two allowed substitution operations. In either case, reversing the interval orientations of edges of R_2 before performing the substitution results in a new interval orientation of the final matrix. Thus, the Δ modules give sets of edges that can be oriented independently of other edges when computing an interval orientation of an interval matrix. The Δ modules are always modules of the interval matrix for the set of intervals that result from the substitution operation. However, not all modules of the interval matrix are Δ modules. An example of this is given in Figure 7. We show below that the Δ modules completely specify the sets of edges that can be oriented independently when computing interval orientations, just as modules do this for transitive orientations.

LEMMA 6.2. *If T is an interval matrix, then for any interval orientation, there is exactly one realizer.*

PROOF. There must be at least one realizer by the definition of an interval orientation. It remains to show that there is only one.

Let a and b be two distinct vertices. If $ab \in G_{1n}$, then $l(a) < l(b)$ and $r(a) < r(b)$ iff (a, b) is the orientation of ab in the interval orientation. If $ab \in G_c$, then $l(b) < l(a)$ and $r(a) < r(b)$ iff $(a, b) \in D_c$. Since the intersection matrix is an interval matrix, $ab \notin G_2$. The relative order of left endpoints is uniquely constrained by D_c and the interval orientation, and so is the relative order of right endpoints.

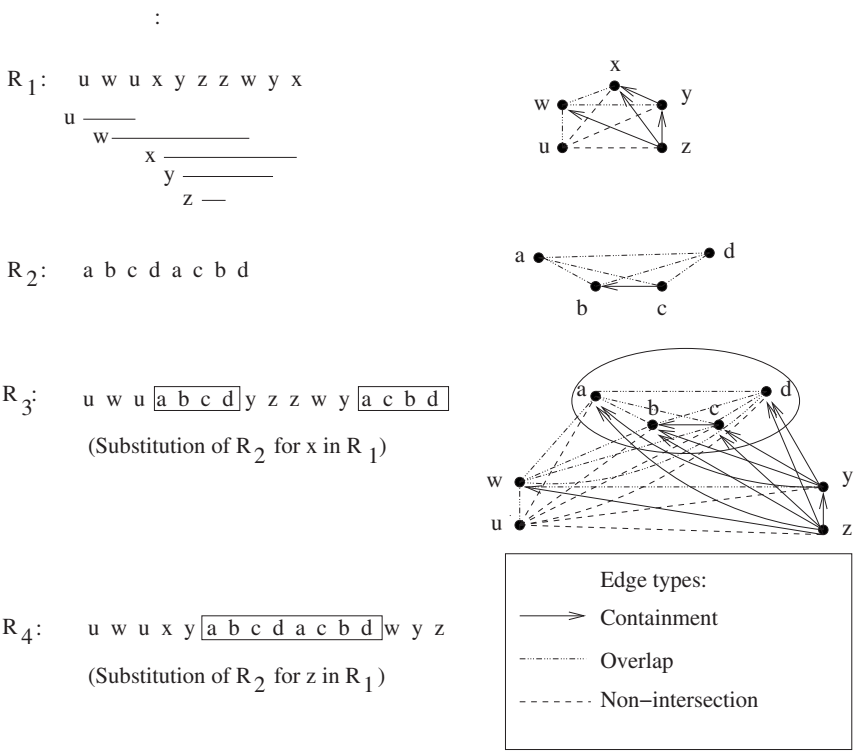


Fig. 6. A Δ module can be viewed as the result of one of two kinds of substitution operations on interval realizers. In these examples, R_2 is substituted for an interval in R_1. R_3 and the picture to its right illustrates the first type, which requires that the left endpoints of R_2 are consecutive and the right endpoints of R_2 are consecutive. The second, illustrated by R_4, requires that the replaced vertex in R_1 has its two endpoints consecutive in the realizer.

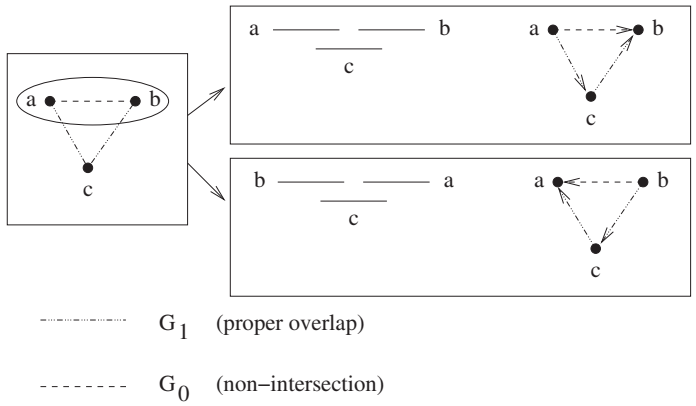


Fig. 7. The pair $\{a, b\}$ cannot be oriented independently of other pairs when computing an interval orientation. There are only two possible interval orientations in this example, so an orientation assigned to $\{a, c\}$ forces the orientation of $\{a, b\}$. The reason for this is that even though $\{a, b\}$ is a module, it fails to be a Δ module.

We now examine the relative order of pairs consisting of one left endpoint and one right endpoint. Let x and y be two vertices. If $x = y$, then $l(x) < r(y)$. Otherwise, suppose without loss of generality that $l(x) < l(y)$. Then $l(x) < r(y)$. In addition, $r(x) < l(y)$ iff $xy \in G_n$, and $l(y) < r(x)$ otherwise. The relative order of all pairs consisting of one left endpoint and one right endpoint are uniquely constrained. □

6.3. *A Γ-Like Relation for Interval Orientations.* Let us define an analog Δ of the Γ relation for the problem of finding an interval orientation instead of just a transitive orientation. (The Γ relation involves two edges joined at one end, resembling the letter Γ, while Δ involves three relationships that make a triangle.) Let Γ_n denote the Γ relation on G_n, let Γ_1 denote the Γ relation on G_1, and let Γ_{1n} denote the Γ relation on G_{1n}.

DEFINITION 6.3. Let $\{a, b, c\}$ be three vertices. Then $(a, b)\Delta(a, c)$ and $(b, a)\Delta(c, a)$ if one of the following applies:

- $(a, b)\Gamma_n(a, c)$ (i.e., $ab, ac \in G_n$ and $bc \in G_{1c}$);
- $(a, b)\Gamma_{1n}(a, c)$ (i.e., $ab, ac \in G_{1n}$ and $bc \in G_c$);
- $ab \in G_n$ and $bc, ac \in G_1$.

We call the last of the conditions in the definition of Δ a *straddle* relationship, since the edges $ac, bc \in G_1$ "straddle" edge $ab \in G_n$.

By analogy to Γ, we let the Δ *implication classes* be the equivalence classes of the transitive symmetric closure of Δ, and the Δ *color classes* be the union of each equivalence class and its transpose.

THEOREM 6.4. *Every interval orientation of G_{1n} in an interval matrix consists of one Δ implication class from each Δ color class.*

PROOF. For every edge $ab \in G_{1n}$, exactly one of (a, b) or (b, a) appears in any interval orientation. An interval orientation is a transitive orientation of G_n, so if $(a, b)\Gamma_n(a, c)$, then both or neither appear in any transitive orientation of Γ_n. It is also a transitive orientation of G_{1n}, so if $(a, b)\Gamma_{1n}(a, c)$, then both or neither appear in any transitive orientation of Γ_{1n}. If $(a, b)\Delta(a, c)$ by the straddle condition, with $ab \in G_n$ and $ac, bc \in G_1$, then a and b contain opposite endpoints of c in any interval realizer. The order of left endpoints must be (a, c, b) or (b, c, a). Either both or neither of (a, b) and (a, c) appear in any interval orientation. □

We now show that Δ has the same dual relationship with the Δ modules as Γ has with modules of a graph. Every Δ module is a module, but some modules are not Δ modules. Like the set of modules, the Δ modules have a decomposition tree where every node is prime or degenerate, and where a set is a member of the family iff it is a node of the tree or a union of children of a degenerate node. The tree is not the modular decomposition tree, however. Just as in the case of Γ, the spans of the equivalence classes induced by Δ are nodes of the tree and pairs of children of degenerate nodes.

DEFINITION 6.5. Let T be an interval matrix. Let $U(T)$ denote the matrix obtained from T by replacing each instance of D_c or $(D_c)^{\mathrm{T}}$ with G_c, thereby "unorienting" the directed edges in D_c. (We define $U(T)$ so that we may avoid dealing with details of the modular decomposition of a matrix that is not symmetric.) We say that a set M of vertices *overlaps* a set E' of edges of T if $T|M$ contains some members, but not all members, of E'. A module of T or of $U(T)$ is a Δ *module* if it is a module that is a clique of $G(T)$, or else a module X such that for no $y \in V - X$, $\{y\} \times X \subseteq G_1$.

It is well known that no module of a graph overlaps a Γ color class [12]. The following is an analogous result about Δ modules and Δ color classes:

LEMMA 6.6. *No Δ module overlaps a Δ color class.*

PROOF. A module M of $U(T)$ is a module of G_n and a module of G_{1n}. Therefore, it cannot overlap a pair of edges that are related by the first two conditions of the definition of Δ. For the third condition, M must be a module of G_1. If it contains $\{a, c\}$, it must contain $\{b, c\}$, which implies that it contains $\{a, b\}$. If it contains $\{a, b\}$, then it is not a clique. If $c \in V - M$, then $\{c\} \times M \subseteq G_1$, and it is not a Δ module of $U(T)$. □

DEFINITION 6.7. A family $\mathcal{F}$ of subsets of a set V is a *tree-decomposable family* if it satisfies the following properties:

1. V and the members of $\{\{x\} : x \in V\}$ are members of $\mathcal{F}$.
2. *Overlap closure*: If X and Y are properly overlapping members of $\mathcal{F}$, then $X \cup Y$, $X \cap Y$, $X - Y$, $Y - X$, and $(X - Y) \cup (Y - X)$ are members of $\mathcal{F}$.

The *strong* members of $\mathcal{F}$ are those members that properly overlap with no other member of $\mathcal{F}$.

THEOREM 6.8 [23], [24], [6], [7]. *If $\mathcal{F}$ is a tree-decomposable family, then the transitive reduction of the containment relation on strong members of $\mathcal{F}$ is a tree. There is a unique way to label the nodes of this tree* prime *and* degenerate *so that a set is a member of $\mathcal{F}$ iff it is a node of the tree or a union of children of a degenerate node.*

We call this tree the *tree decomposition* of $\mathcal{F}$. The modular decomposition of an undirected graph or a symmetric matrix is just the tree decomposition of its modules, which are a tree-decomposable family.

THEOREM 6.9. *The Δ modules of $U(T)$ are a tree-decomposable family.*

PROOF. The modules of a symmetric matrix are a tree-decomposable family [6], [7]. Let X and Y be overlapping Δ modules. Since they are modules, $X \cup Y$, $X \cap Y$, $X - Y$, $Y - X$, and $(X - Y) \cup (Y - X)$ are modules. If X and Y are both cliques of $G(T)$, then vertices in $X - Y$ have edges in $G(T)$ to $X \cap Y$, hence to all of Y. $X \cup Y$ is a clique, so all modules that are subsets of $X \cup Y$ are Δ modules, and the claim holds.

We therefore assume in the remainder of the proof that X is not a clique.

We now show that $X \cap Y$ is a Δ module. If $X \cap Y$ is a clique, then, since it is a module, it is a Δ module. Otherwise, neither X nor Y is a clique. Since both of these sets are Δ modules, there exists no G_1 edges from members of X to vertices outside of X, or G_1 edges from members of Y to vertices outside of Y. An edge from a vertex of $X \cap Y$ to a vertex outside of $X \cap Y$ either goes from a member of X to a nonmember of X, or from a member of Y to a nonmember of Y. Therefore, there is no edge of G_1 from $X \cap Y$ to a vertex outside $X \cap Y$. Since $X \cap Y$ is a module, it must be a Δ module.

Next, we show that $X - Y$ is a Δ module. If $X - Y$ is a clique, it is a Δ module, since it is a module. If $X - Y$ is not a clique, then since X is a Δ module, any $w \in V - (X - Y)$ such that $\{w\} \times (X - Y) \subseteq G_1$ must reside in $X \cap Y$. Suppose such a w exists. Since w resides in Y and Y is a module, $(X - Y) \times Y \subseteq G_1$. For $w' \in Y - X$, $\{w'\} \times X \subseteq G_1$, a contradiction, since X is a Δ module. The same analysis applies to $Y - X$.

Finally, we show that $U = (X - Y) \cup (Y - X)$ is a Δ module. If U is a clique, it a Δ module, since it is a module. Otherwise, any vertex $w \in V - U$ such that $\{w\} \times U \subseteq G_1$ must lie inside X, since X is a Δ module and not a clique. Therefore, $w \in X \cap Y$. Since X is a module and $(Y - X) \times \{w\} \subseteq G_1$, $(Y - X) \times X \subseteq G_1$. $(X - Y) \cup (Y - X)$ can only fail to be a clique if $X - Y$ or $Y - X$ fails to be a clique. Suppose $X - Y$ fails to be a clique. Then X fails to be a clique. $(Y - X) \times X \subseteq G_1$ contradicts the assumption that X is a Δ module. □

DEFINITION 6.10. We call the tree decomposition of $U(T)$ the *Delta tree* of $U(T)$, and denote it $\Delta(U(T))$.

Just like the modular decomposition, let the $\Delta(U(T))$ tree be represented in $O(n)$ space by creating a node of size $O(1)$ to represent each node of $\Delta(U(T))$, giving each node a label to indicate whether it is prime or degenerate as well as a list of pointers to its children in the transitive reduction of the containment relation. The quotient induced by children of a degenerate node in their parent is complete in G_n, G_1, or G_c, so each degenerate node may be labeled according to which of these cases applies. The set U represented by a node can be retrieved in $O(|U|)$ time by visiting its leaf descendants, so there is no advantage to labeling a node with a list of its members. Figure 8 gives an example.

Theorems 6.11–6.13 show that Δ modules satisfy the remaining properties of a class of set system described by Möhring in [23] and [24].

THEOREM 6.11. *If X is a Δ module of $U(T)$ and Y is a subset of X, then Y is a Δ module of $U(T)$ iff it is a Δ module of $U(T)|X$.*

PROOF. If Y is a clique, the claim holds because of the corresponding theorem about modules of a matrix [6], [7]. If Y is not a clique, then neither is X, and since X is a Δ module, there exists no $y \in V - X$ with an edge of G_1 to any member of X, hence to any member of y. Then Y fails to be a Δ module in $U(T)$ iff there exists $y \in X - Y$ with G_1 edges to Y, which also determines whether it is a module of $U(T)|X$. □

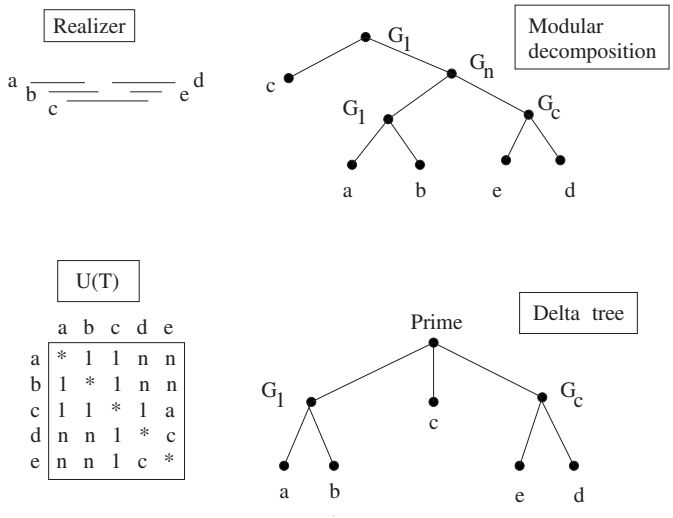


Fig. 8. An interval realizer, $U(T)$, the modular decomposition of $U(T)$, and the Δ tree $\Delta(U(T))$.

THEOREM 6.12. *If X is a Δ module of $U(T)$ and $Y \subseteq V$ intersects X, then $X \cap Y$ is a Δ module of $U(T)|Y$.*

PROOF. The analogous theorem for modules of a matrix is given in [6] and [7]. Thus, $X \cap Y$ is a module of $U(T)|Y$. X can only fail to be a Δ module of $U(T)|Y$ if $X \cap Y$ is not a clique and there exists $w \in Y - X$ such that $\{w\} \times (X - Y) \in G_1$. However, this would imply that X is not a clique in $U(T)$ and $\{w\} \times X \in G_1$ in $U(T)$, contradicting X's status as a Δ module in $U(T)$. □

THEOREM 6.13. *If $\mathcal{P}$ is a modular quotient on $U(T)$ consisting of Δ modules of $U(T)$, and $\mathcal{P}' \subseteq \mathcal{P}$, then $\mathcal{P}'$ is a Δ module of $U(T)/\mathcal{P}$ iff $\bigcup \mathcal{P}'$ is a Δ module of $U(T)$.*

PROOF. By Lemma 6.6, if $\bigcup \mathcal{P}'$ is a Δ module, then there is no straddle relationship containing edges of $U(T)|\bigcup \mathcal{P}'$ and other edges. By Theorem 6.12, applied to the submatrix induced by one representative from each member of $\mathcal{P}$, this is also true of $\mathcal{P}'$ in $U(T)/\mathcal{P}$. On the other hand, if $\mathcal{P}'$ is a Δ module in $U(T)/\mathcal{P}$, then there exists $y \in V - \bigcup \mathcal{P}'$ such that $\{y\} \times \bigcup \mathcal{P}' \in G_1$ iff this is true for $\mathcal{P}'$ in $U(T)/\mathcal{P}$. If there exists such a y, then $\mathcal{P}'$ is a clique of $U(T)/\mathcal{P}$, and $\bigcup \mathcal{P}'$ can only fail to be a module if it fails to be a clique. This would imply that there is some $X \in \mathcal{P}'$ that is not a clique. However, because $y \in V - X$ and $\{y\} \times X \subset G_1$, this would contradict the assumption that X is a Δ module. □

LEMMA 6.14. *The set of vertices spanned by a Δ color class in an interval matrix T is a Δ module of $U(T)$.*

PROOF. G_2 is empty in T, since T is an interval matrix. Let S be the set of vertices spanned by a color class C. Suppose that S is not a module. Then there exists $x \in V - S$ that has different relationships in T to members of S. C is connected, as only coincident edges are related by Δ. Therefore, x has different relationships between two vertices $y, z \in S$ such that $yz \in C$. At least one edge from x to $\{y, z\}$ is an edge of G_{1n}; without loss of generality, suppose that $xy \in G_{1n}$. If $xz \in G_c$, then $(x, y)\Gamma_{1n}(z, y)$, and $xy \in C$ contradicting x's nonmembership in S.

Therefore, xy and xz are edges of G_{1n}. Without loss of generality, suppose $xy \in G_1$ and $xz \in G_n$. If $yz \in G_n$, then $(x, z)\Gamma_n(y, z)$, and we again get the contradiction $x \in S$. If $yz \in G_1$, then $(y, z)\Delta(x, z)$ by the straddle rule, and again we get the contradiction $x \in S$. Since x does not exist, we conclude that S is a module.

Suppose that S is a module that is not a Δ module. Then it is not a clique, and there exists $w \in V - S$ such that $\{w\} \times S \subseteq G_1$. There exist $s_1, s_2 \in S$ such that $s_1 s_2 \in G_n$.

Suppose $s_1 s_2 \in C$. Then $(w, s_1)\Delta(s_2, s_1)$, w is among the vertices spanned by C, hence $w \in S$, a contradiction. We conclude that C consists exclusively of edges of G_1, and $s_1, s_2 \in S$ such that $s_1 s_2 \in G_n$ and $s_1 s_2$ is in a different color class C'. Let $P = (s_1 = x_1, x_2, \ldots, x_k = s_2)$ be a simple path from s_1 to s_2 whose edges are drawn from C.

To obtain a contradiction with $s_2 x_{i-1} \in G_1$, we show by induction that for each i from 1 to $k - 1$, $s_2 x_i \in G_n$ and $s_2 x_i \in C'$. This is true for $i = 1$, since $s_2 s_1 \in G_1$ and $s_2 s_1 \in C'$. Suppose by induction that $i > 1$ and that it is true for $i - 1$. Then $x_{i-1}x_i \in G_1$, $x_{i-1}x_i \in C$, $s_2 x_{i-1} \in G_n$, and $s_2 x_{i-1} \in C'$. If $s_2 x_i \in G_c$ or $s_2 x_i \in G_1$, then $x_{i-1}x_i \Delta x_{i-1}s_2$, contradicting $C' \neq C$. Thus, $s_2 x_i \in G_n$, and $(s_2, x_i)\Delta(s_2, x_{i-1})$, hence $s_2 x_i$, like $s_2 x_{i-1}$, is an element of C'. □

THEOREM 6.15. *A set of edges of G_{1n} is a Δ color class iff it is the set of edges of G_{1n} connecting children of a prime node in the Δ tree of $U(T)$ or the set of edges of G_{1n} connecting a pair of children of a degenerate node. If X and Y are two children of a node of the tree, then no Δ implication class contains both directed edges in $X \times Y$ and directed edges in $Y \times X$.*

PROOF. Let uw be an edge of G_{1n}. If uw connects two children of a prime node A, then all Δ modules that contains u and w contain A. The color class C containing uw must span A by Lemma 6.14. C cannot span a larger set or contain edges that are internal to a child of A, by Lemma 6.6, and the claim follows for C. If uw connects two children B_1 and B_2 of a degenerate node, an identical argument applies.

For the claim about the orientations of edges between X and Y, suppose first that X and Y are two children of a degenerate node Z. If Z is a G_c node, the claim is vacuous. Otherwise, suppose that $a, b \in X$ and $c \in Y$ such that (a, c) and (c, b) are in one implication class. Let A and B be the partition of X such that $A \times \{c\}$ and $\{c\} \times B$ are in the same implication class. If Z is a G_n node, then by transitivity of orientations of G_n in interval orientations, $A \times B \subset G_n$, and $B \cup Y$ is a Δ module of $U(T)$, contradicting X's status as a strong Δ module. If Z is a G_1 node, then X and Y are cliques of $G(T)$. By transitivity of interval orientations of G_{1n}, $A \times B \subseteq G_1$, and $B \cup Y$ is a Δ module, which is again a contradiction. If Z is Δ prime, that is, it has only trivial Δ modules, then by Theorem 6.13, the submatrix induced by one representative vertex from each child of Z

has only trivial modules. The orientation of an edge of this substructure cannot be reversed in an interval orientation without reversing all edges in it. If a is the representative of X and c is the representative of Y, then replacing a with b gives an interval orientation of an isomorphic submatrix that differs only in the orientation of a single edge, a contradiction. □

LEMMA 6.16. *Let T be an intersection matrix. Let x be a vertex, let M be a maximal module of $U(T)$ that does not contain x, and let $y \notin M$. Then there exists a sequence $(x = x_1, x_2, \ldots, x_k = y)$, such that for each i from 1 to $k-1$, $x_i \in V - M$ and has a different relationship to x_{i+1} from the relationship it has to M.*

PROOF. We give a procedure to construct such a sequence. If $x = y$, the sequence is (x). Otherwise, let $A = V - \{x\}$. Since M is a maximal module that does not contain x, it follows that while $A \neq M$, there exists some vertex $w \notin A$ that has different relationships to the members of A. Partition A into maximal groups of vertices that have the same relationship to w. Since M is a module, M is a subset of a resulting part P of A. Reset $A := P$, and iterate, halting when $A = M$.

We now show how to construct the required sequence from right to left. Since $y = x_k \notin M$, let x_{k-1} be the vertex w that caused y to separate from members of M. Since x_{k-1} was not a member of A at that time, there is another vertex, x_{k-2} that served as w when x_{k-1} separated from M. If $x_{k-2} \neq x$, there is another vertex x_{k-3} that served as w when x_{k-2} was split apart from M. Iterating, we eventually halt with $x_{k-(k-1)} = x$, having constructed the sequence $(x_1, x_2, \ldots, x_k)$. □

LEMMA 6.17. *Let x be a source or sink in some interval orientation of G_{1n}, and let $\mathcal{P}$ denote $\{x\}$ and the maximal modules of $U(T)$ that do not contain x. Then every member of $\mathcal{P}$ is a Δ module.*

PROOF. Suppose M is a member of $\mathcal{P}$ that is not a Δ module. Then M is not a clique of $G(T)$, and there exists $y \in V - M$ such that $\{y\} \times M \subseteq G_1$. Let s_1, s_2 be two nonadjacent vertices of M, and assume that s_1's right endpoint precedes s_2's left endpoint in a realizer where x is a source or sink.

By Lemma 6.16, there exists a sequence $(x = x_1, x_2, \ldots, x_k = y)$ such that for each i from 1 to $k-1$, $x_i \in V - M$ and has a different relationship to x_{i+1} from the relationship it has to M.

Next, we show by induction from right to left in this sequence that for each x_i, $[l(x_i), r(x_i)]$ is contained in $[l(s_1), r(s_2)]$. This is true for $x_k = y$, since s_1 and s_2 are disjoint intervals and y properly overlaps each of them. Suppose that it is true for x_i, where $i \leq k$. Since M is a module, x_{i-1} has the same relationship to s_1 and s_2, and so does x_i. If x_{i-1} has a G_1 relationship to s_1 and s_2, then, like x_k, x_{i-1} is contained in $[l(s_1), r(s_2)]$. It cannot be the case that x_{i-1} is contained in both intervals s_1 and s_2, since they are disjoint. If x_{i-1} contains both intervals, then it contains x_i also, and has the same relationship to all three, contradicting the definition of the sequence $(x_1, \ldots, x_k)$. If it has a G_n relationship to s_1 and s_2, then, since it must intersect x_i in order to have

a different relationship to it, interval x_{i-1} lies between intervals s_1 and s_2. In any case, $[l(x_{i-1}), r(x_{i-1})]$ is contained in $[l(s_1), r(s_2)]$.

We conclude that interval $x = x_1$ is contained in $[l(s_1), r(s_2)]$. It is neither contained in s_1 and s_2 nor contains them, so its relationship to them is in G_{1n}. However, then s_1 is a predecessor of x and s_2 is a successor of x in the corresponding interval orientation of G_{1n}. Therefore, x cannot be a source or source or sink in any interval orientation of G_{1n}, a contradiction. □

LEMMA 6.18. *Let T be an interval matrix, let R be an interval realizer of T, and let D_{1n} be the interval orientation assigned to G_{1n} by R.*

1. *If Y is a node of $\Delta(T)$ that is a clique of $G(T)$, the left endpoints of Y are consecutive in R and the right endpoints of Y are consecutive in R.*
2. *If Z is a node of $\Delta(T)$ that is not a clique, the endpoints of Z are consecutive in R.*

PROOF. The lemma follows almost immediately from Theorem 6.15. For part 1, the left endpoints of Y precede the right endpoints of Y, since it is a clique. Let $x \in V - Y$. If $\{x\} \times Y \subseteq D_c$, then the endpoints of x lie between the left endpoints of Y and the right endpoints of Y. If $\{x\} \times Y \subseteq (D_c)^{\mathrm{T}}$, then the left endpoint of x precedes all endpoints of Y, and the right endpoint of x follows all endpoints of Y. If $\{x\} \times Y \subseteq G_n$, then x is disjoint from all intervals in Y. If $\{x\} \times Y \subseteq G_1$, then, by Theorem 6.15, one endpoint of x lies in the common intersection of intervals in Y, and the other is disjoint from them; otherwise the interval orientation would contain a mixture of directed edges from x to Y and from Y to x. Since x is an arbitrary member of $V - Y$, no endpoint of a nonmember of Y appears among left endpoints of Y or among right endpoints of Y.

For part 2, if Z is not a clique, then it has no incident overlap edges, since it is a node of $\Delta(U(T))$. If there existed $x \in V - Z$ such that $\{x\} \times Z \subseteq D_c$, this would force all intervals in Z to contain both endpoints of x, making Z a clique. Thus, for every $x \in V - Z$, $Z \times \{x\} \subseteq D_c$ or $Z \times \{x\} \subseteq G_n$. In either case, x does not have any endpoints between two endpoints of Z. □

THEOREM 6.19. *Any acyclic union of implication classes gives an interval orientation of G_{1n} in T.*

PROOF. If W is a node of the Delta tree, then by Lemma 6.18, the endpoints of intervals realizing W in any interval realizer can be replaced with their mirror transpose. This reverses the orientations of edges of G_{1n} in the interval orientation that the realizer gives. Repeating this operation on the children leaves the net effect of reversing the orientations of edges that go between children of W.

If A and B are two consecutive children in the interval orientation that a realizer assigns to a degenerate node C labeled G_n or G_1, then either all endpoints of A are consecutive and followed immediately by the consecutive endpoints of B, or the consecutive left endpoints of A are followed immediately by the consecutive left endpoints of B, and the consecutive right endpoints of A are followed immediately by the consecutive right endpoints of B. The orientation of edges between A and B can be inverted without

affecting other orientations, by swapping the relative order of these groups of endpoints. By a sequence of such swaps, an arbitrary permutation of children of C can be induced.

By Theorem 6.15, all acyclic unions of implication classes can be obtained by a series of these swaps. By Lemma 6.6, this gives all interval orientations of G_{1n} in T. □

Johnson and Spinrad have independently developed an idea that is related to the Δ tree [15]. Their tree, which they call an MD-PQ tree, has $2n$ leaves instead of n leaves, and each leaf represents an endpoint of an interval in a realizer of G, rather than a vertex. The set of permutations of the leaves represented by a set of allowable reorderings of children of nodes on the tree represents all realizers of G, rather than realizers of an intersection matrix.

6.4. *An Algorithm for Finding a Realizer of an Interval Matrix.* The approach works by finding an interval orientation of G_{1n} and then finding the corresponding interval realizer. This requires us to give solutions to the following:

PROBLEM 1. Given a topological sort of an interval orientation of G_{1n}, find the corresponding interval realizer.

PROBLEM 2. Find a topological sort of an interval orientation of G_{1n}.

Our algorithm is a variant of the transitive orientation algorithm of [18]. That algorithm uses the Γ relation to constrain the orientation it produces. We adapt the algorithm to use the Δ relation instead of the Γ relation to constrain the orientation.

6.4.1. *Problem* 1: *given an interval orientation, find the corresponding interval realizer.* Given an interval orientation of an interval matrix in the form of a topological sort of the orientation of G_{1n}, it is easy to find the corresponding interval realizer in linear time. The union of D_c and the orientation of G_{1n} is a linear order on the vertices, and gives the order of right endpoints of the intervals, since if x is a vertex, the vertices with earlier right endpoints are those that are either predecessors in the orientation of G_{1n} or vertices whose interval is contained in x's interval. Similarly, the union of $(D_c)^{\mathrm{T}}$ and the orientation of G_{1n} is also a linear order, and gives the order of left endpoints of the intervals.

Let P be the given linear extension of the interval orientation. To find the left-endpoint order L of vertices given by the union of $(D_c)^{\mathrm{T}}$ and the interval orientation of G_{1n}, we find the position of each vertex x in L. This is obtained by adding up the number n_c of predecessors of x in D_c, the number n_1 of neighbors of x in G_1 that are predecessors in the orientation of G_{1n}, and the number n_n of neighbors of x in the orientation of G_n that are predecessors in the orientation of G_n. The first two of these can be easily found in $O(1 + N(G(T), x))$ time. Let $p(x)$ be the position of x in P. We find n_n by evaluating $n_n = (p(x) - 1) - (n_c + n_1)$. Doing this for all vertices takes time proportional to the sum of degrees of the vertices in $G(T)$, or $O(n + m)$ time. This is essentially the trick used in [18] for finding a realizer of a permutation graph, given transitive orientations of the graph and its complement.

We can create the full realizer by zipping these two permutations together in a way that reflects the adjacencies of each vertex. Let $v_1, v_2, \ldots, v_n$ be the vertices in left-to-right order of right endpoint. We place the right endpoint among the left endpoints $(l(v_1), l(v_2), \ldots, l(v_n))$, starting with $r(v_1)$, and working up through $r(v_n)$, placing $r(v_i)$ in the first position that is both to the right of $r(v_{i-1})$ and to the right of the rightmost left endpoint of neighbors of v_i.

6.4.2. *Overview of solution to Problem* 2: *finding an interval orientation of an interval matrix.* Recall that we are required only to give a topological sort of G_{1n}. This saves us from having to orient edges of G_n explicitly, which would violate the time bound.

The basic operation begins with a partition $\mathcal{P}$ of the vertices and refines the partition classes until every partition class is a Δ module. It does this in a way that avoids splitting apart members of any Δ module that is initially a subset of a class of $\mathcal{P}$. The final partition therefore gives the maximal Δ modules that were initially subsets of classes of $\mathcal{P}$.

Given a partition $\mathcal{P}$ of the vertices, we define an *unordered pivot* on partition class X with pivot vertex $w \notin X$ to be the refinement of $\mathcal{P}$ obtained as follows. The first type of pivot is a *standard pivot*, and is applied when X fails to be a module of $U(T)$. Let X_c be the members of X whose relationship to w is an edge of G_c, let X_1 be those whose relationship to w is an edge of G_1, and let X_n be those whose relationship to w is an edge of G_n. Remove X from $\mathcal{P}$ and replace it with the nonempty members of $\{X_c, X_1, X_n\}$.

For instance, suppose $\mathcal{P} = \{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}, \{v_7\}\}$, $X = \{v_4, v_5, v_6\}$, the pivot vertex is v_2, $X_c = \{v_4\}$, $X_1 = \emptyset$, and $X_n = \{v_5, v_6\}$. After the pivot, $\mathcal{P} = \{\{v_1, v_2, v_3\}, \{v_4\}, \{v_5, v_6\}, \{v_7\}\}$.

There is another type of pivot, a *modular pivot*, that is used only when X is a module of $U(T)$ that fails to be a Δ module. This is used in this case because a standard pivot cannot split X, even though it fails to be a Δ module. Let w be a vertex in $V - X$ such that the vertices in X are neighbors in G_1. Since X is not a Δ module, such a w exists and X is not a clique. Let x be a vertex with an incident edge in $G_n|X$, and let Y be the neighbors of x in $G_n|X$. The modular pivot consists of replacing X in $\mathcal{P}$ with Y and $X - Y$.

It is not hard to see that if a standard pivot applies to X, any Δ module that is a subset of X is a subset of X_c, X_1, or X_n, and that if a modular pivot applies, any Δ module that is a subset of X is a subset of Y or of $X - Y$. It follows that the partition can be refined until every partition class is a Δ module, and at this point, the partition gives the maximal Δ modules that were subsets of a single partition class of the initial partition.

Algorithm 6.20 gives a key procedure for our approach, `UnorderedPartition`, which is based on iterated pivots.

ALGORITHM 6.20. Refine a partition with iterated pivots.

```
UnorderedPartition (T, v)
  P := {{v}, V − {v}}
  While there exists Y ∈ P that is not a Δ module
    Perform an unordered pivot that splits Y
  Return P
```

We now modify `UnorderedPartition` so that it maintains an ordering on the partition classes of $\mathcal{P}$ as it refines it. Initially, the ordering is $(\{v\}, V - \{v\})$.

When X is split with a pivot on w, we keep track of the spot formerly occupied by X in the linear order. When X_c, X_1, and X_n are inserted, they are inserted consecutively at that spot. Their relative order within that spot is determined as follows. If w lies in a partition class that precedes X's spot, then their relative order is (X_c, X_1, X_n); otherwise it is (X_n, X_1, X_c). In the case of a modular pivot, we place $X - Y$ first if the pivot w resides in a class that is later in the ordering than X, and place it second if the pivot resides in an earlier class.

We may call this ordered variant `Partition`. Henceforth in the paper we refer to this (ordered) pivot when we use the term "pivot."

We say an ordering of partition classes on V is *consistent* with an interval orientation if all arcs in the orientation that go between partition classes are oriented from earlier to later partition classes in the order.

To explore the key insights in a simple setting, let us suppose that T is an interval matrix and $U(T)$ is Δ prime. Suppose also that we know a vertex v that is a source in some interval orientation D_{1n} of G_{1n}. Then the initial ordered partition $(\{v\}, V - \{v\})$ is consistent with D_{1n}. Moreover, it is easy to see by induction on the number of pivots that every successive refinement is consistent with the interval orientation. The reason for this is that after a standard pivot, the set E' of edges of G_{1n} that go between members of $\{X_c, X_1, X_n\}$ have a Δ relationship to the set E'' of edges that are incident to both X and the pivot vertex w. By induction, edges in E'' are all oriented toward X in D_{1n} if w is in an earlier class than X in the ordering on $\mathcal{P}$, or all oriented toward w if w is in a later class. Consider the relative ordering of $\{X_c, X_1, X_n\}$ as (X_c, X_1, X_n) or (X_n, X_1, X_c). The Δ relationships between E' and E'' force members of E' to be oriented to the right in this ordering, which means that the induction hypothesis remains true after the pivot.

For instance, suppose that w precedes X. Let ab be an edge of G_{1n} such that $a \in X_c$ and $b \in X_1$. Then $wa \in G_c$, $wb \in G_1$, and $ab \in G_{1n}$, hence $(a, b)\Delta(w, b)$. Since w precedes b, we may assume by induction that (w, b) is in D_{1n}, so (a, b) is also in D_{1n}.

A similar analysis applies when $a \in X_1$, $b \in X_n$, or $a \in X_c$ and $b \in X_n$. We show below that modular pivots also maintain the invariant that the ordered partition remains consistent with the interval orientation.

Since $U(T)$ is Δ prime, it has only trivial modules. Thus, all partition classes are singletons when the procedure halts. Since the ordering on these is consistent with an interval orientation, their ordering is a topological sort of D_{1n}. This topological sort, together with the representation of G_{1n} given by T, gives a representation of D_{1n} that is adequate for our purposes.

To get this solution, we assumed above that v was a source in an interval orientation in order to ensure that the induction hypothesis was true initially. It remains to establish how to find v. Somewhat surprisingly, v may be found by the same procedure. Selecting an arbitrary vertex u, and starting `Partition` with initial partition $(\{u\}, V - \{u\})$ yields an ordering on the vertices as before. The last vertex v in this ordering is a source in any interval orientation that orients vu as (v, u). This is shown by a similar induction. The induction hypothesis this time is that as the partition is refined, all edges of G_{1n} from v to vertices outside of v's current partition class in $\mathcal{P}$ are oriented away from v. This is true initially when the only such edge is vu.

By these observations, Algorithm 6.21 gives a procedure for producing a topological sort of an interval orientation when $U(T)$ is prime.

ALGORITHM 6.21. Find a topological sort of an interval orientation of G_{1n} when $U(T)$ is prime.

OrientPrime (T)
  Select arbitrary vertex u
  $\mathcal{P}_1$ = `Partition` (T, u)
  Let v be the member of the rightmost class in $\mathcal{P}_1$
  $\mathcal{P}$:= `Partition` (T, v)
  Return $\mathcal{P}$ as the topological sort

We now relax the assumption that $U(T)$ is Δ prime and generalize this procedure to obtain a general solution to our problem. `Partition(`T, v`)` has no way to break apart any Δ modules of $U(T)$ that are contained in $V - \{v\}$. Instead, it may halt with some partition classes that are nonsingleton Δ modules. Thus, the final ordering of partition classes leaves unspecified the order on the vertices that are contained inside such a module.

The way we get around this is to start up the partitioning process recursively inside such a Δ module in order to complete the ordering. The algorithm then makes two calls to this recursive variant, `RPartition`, in place of two calls to `Partition`. Algorithm 6.22 gives the algorithm, with `RPartition` implemented as an iterative procedure that makes a series of calls to `Partition`.

ALGORITHM 6.22. `Orient(T)` finds a topological sort of an interval orientation of an interval matrix T.

RPartition (T) // vertices have been numbered
  Let $\mathcal{P} = (V)$
  While not every partition class in $\mathcal{P}$ is a singleton set
    Let Z be a nonsingleton class of $\mathcal{P}$
    Let v be the highest-numbered vertex in Z
    $\mathcal{Z}$:= `Partition` $(T|Z, v)$
    Substitute $\mathcal{Z}$ for Z in the ordering on $\mathcal{P}$
  Return the ordering of V given by $\mathcal{P}$

`Orient` (T)
  Number the vertices in any order
  Call `RPartition` (T) to get an ordering of the vertices
  Renumber the vertices in left-to-right order in this ordering
  Run `RPartition` (T) to get a new ordering of the vertices
  Return this ordering as the topological sort

We show below that in the second call to `RPartition` from `Orient`, the highest-numbered vertex in Z in the loop is a source in an interval orientation of $G_{1n}|Z$. We also show that Z is a Δ module. Therefore, we do not need to worry about interactions

between orientations of edges of $G_{1n}|Z$ and other edges of G_{1n} in assigning an interval orientation to them; the call to `Partition(`$T|Z, v$`)` does not need to take the rest of $\mathcal{P}$ into account in maintaining the invariant that the ordering on $\mathcal{P}$ is consistent with an interval orientation. The procedure halts when all partition classes are singletons, and at this point, the order on them will be a topological sort of an interval orientation of G_{1n}.

6.4.3. *Proof of correctness of Algorithm* 6.22

LEMMA 6.23. *Let X, Y, x, and w be as in the definition of a modular pivot. All edges of G_{1n} in $\{Y \times (X - Y)\}$ are in the same Δ implication class as (w, x).*

PROOF. All members of $Y \times \{x\}$ are edges of G_n. Since all members $\{w\} \times (Y \cup \{x\})$ are edges of G_n, all edges of $Y \times \{x\}$ are in the Δ relation with (w, x).

Let yq be an arbitrary edge of G_{1n} such that $y \in Y$ and $q \in X - Y$. If $q = x$, the foregoing shows that (y, q) is in the same implication class as (w, x). Otherwise, note that $xq \in G_{1n}$; otherwise $q \in Y$, a contradiction, so $yq \in G_{1n}$ and $yx \in G_n$ yields $(y, q)\Delta(y, x)$, and since $(y, x)\Delta(w, x)$, the result follows. □

LEMMA 6.24. *Let T be an interval matrix.*

1. *If a sequence of partition classes on vertices of T is consistent with some interval orientation D of G_{1n} in T, then they remain consistent with D after a pivot.*
2. *Let X be the rightmost class in a sequence of partition classes, let X' be the rightmost class after the next successful standard pivot on a rightmost subset of X and let $q \in X'$. In any interval orientation D' of G_{1n} where all edges of D' between $V - X$ and $\{q\}$ are oriented toward q, all edges of D' between $V - X'$ and $\{q\}$ are also oriented toward q in D'.*

PROOF. For part 1, suppose that the claim of part 1 applies before a pivot on w is used to split a partition class X. Suppose w occurs in a partition class that is earlier than X in the ordering; the analysis is similar if it occurs in a later partition class.

If the pivot is a modular pivot, the result is immediate from Lemma 6.23. If the pivot is a standard pivot, let ab be an edge of G_{1n} such that a occurs in a class that is earlier in the sequence (X_c, X_1, X_n) than b does. If $a \in X_c$ and $b \in X_1 \cup X_n$, then $(a, b)\Gamma_{1n}(w, b)$, implying $(a, b)\Delta(w, b)$. Suppose $a \in X_1$ and $b \in X_n$. If $ab \in G_n$, then $(a, b)\Gamma_n(w, b)$, implying again that $(a, b)\Delta(w, b)$, and if $ab \in G_1$, then $(a, b)\Delta(w, b)$ by the straddle condition. Therefore (a, b) is also an edge in the interval orientation. Since ab is an arbitrary edge that goes between members of $\{X_c, X_1, X_n\}$, the invariant is maintained.

For part 2, suppose the next successful pivot on X is a standard pivot. Then it is easy to see that any edge of G_{1n} from q to a class of X_c, X_1 that does not contain q has the Δ relationship to (q, w), and the result follows.

Otherwise, the next pivot is a modular pivot. Then X is a Δ module, as modular pivots are only used in this case. Let w, x, and Y be as in the definition of the modular pivot. Since X is rightmost, w is in an earlier partition class, so after the pivot, $X - Y$ is the rightmost partition class. If there is an edge yq in G_n for $y \in Y$, then $wq, wy \in G_1$ and $yq \in G_n$ imply $(w, q)\Delta(y, q)$. Since $(y, q) \in Y \times (X - Y)$, it follows from Lemma 6.23 that $(w, q) \in D'$ implies that every edge of G_{1n} in $Y \times \{q\}$ is in D', and the result follows.

The remaining case is when q does not have any element of Y as a neighbor in G_n. Since X is a Δ module, the next successful pivot on $X - Y$ will leave x in the rightmost class and remove q from the rightmost class, so q is irrelevant to the claim. □

LEMMA 6.25. *If M is a Δ module of $U(T)$, the presence of vertices in $V - M$ has no effect on the relative ordering of members of M produced by* `RPartition`.

PROOF. In each call, if a and b are two members of M, they are initially in the same partition class, V, and in the end they are in two different partition classes, $\{a\}$ and $\{b\}$. The event that first splits them into two classes is either the replacement of Z with $(\{v\}, Z - \{v\})$ at the beginning of a call to `Partition`, or a pivot on some vertex w. In the former case the presence of nonmembers of M has no effect on the relative order of $\{a\}$ and $\{b\}$, since this depends only on the order of the numbers assigned to a and to b. In the latter case, $w \in M$, a pivot outside of M cannot split a partition class that is a subset of M. □

LEMMA 6.26. *Let M be a Δ module of $U(T)$.*

1. *After the first call to* `RPartition` *from* `Orient`, *the rightmost vertex in M is a sink in an interval orientation of $G_{1n}|M$.*
2. *If M is a Δ module of $U(T)$, then after the second call to* `RPartition` *from* `Orient`, *the ordering of vertices in M is a linear extension of an interval orientation of $G_{1n}|M$.*

PROOF. For part 1, suppose by induction that the claim is true for Δ modules that are proper subsets of M. M is first split up when a single vertex of M is removed from a partition class Z that contains M. `Partition` returns $\mathcal{Z}$, where all members of $\mathcal{Z}$ are Δ modules of $U(T)$. For each $X \in \mathcal{Z}$ that contains members of M, $X \cap M$ is a Δ module in $U(T)|M$, by Theorem 6.12, and collectively, these intersections partition M into Δ modules of $U(T)|M$. By Lemmas 6.25 and 6.24(part 2), and induction on the number of pivots in the while loop of `Partition`, the rightmost class M' in this partition of M is a sink with respect to edges between $M - M'$ and M'. By Lemma 6.25 and the assumption that part 1 is true for smaller modules than M, the subsequent partition of M' places a sink s in an interval orientation of $G_{1n}|M'$ in the last position of the ordering it returns. Since M' is a Δ module, there can be no Δ constraints between edges of $G_{1n}|M'$ and edges of $G_{1n}|M$ that are not edges of $G_{1n}|M'$, by Lemma 6.6. There is an acyclic union of implication classes where s is a sink, by Theorem 6.19.

We now consider the second claim. M is first split up when a source in an interval orientation of $T|M$ is split off from M. The partition $\mathcal{Z}$ returned by `Partition` consists of Δ modules, by Lemma 6.17. Since M is a Δ module, the partition $\mathcal{M}$ that they induce in M consists of Δ modules of $T|M$, by Theorem 6.9. The ordering of $\mathcal{M}$ is consistent with an interval orientation of $T|M$, by Lemmas 6.24 and 6.25. By induction on the size of a Δ module, the remaining pivots produce interval orientations of each member of $\mathcal{M}$, and since they are Δ modules, these orientations can be combined with the ordering on $\mathcal{M}$ to give an interval orientation of $T|M$, by Theorem 6.19. □

Since V is a Δ module, the second call to `RPartition` produces a linear extension of an interval orientation. Once `RPartition` is run, an interval realizer may be constructed

and checked in linear time, using Algorithm 5.1. The bottleneck for the time bound is the running time of `RPartition`.

6.4.4. *An $O(n + m \log n)$-time implementation.* We have already shown that the time to construct a realizer of an interval matrix is given by the running time of `RPartition` on the matrix. The first call to `RPartition` in Algorithm 6.22 serves only to partition the rightmost class in the ordering. Performing only pivots on the rightmost class until it is a singleton set takes $O(n + m)$ time, since it requires one pivot on each vertex after it leaves the rightmost class. The second call to `RPartition` does not require any modular pivots, by Lemma 6.17. Thus, in obtaining the time bound, we may focus only on the second call and assume that all pivots are standard pivots.

Algorithm 6.27 gives an implementation of the `while` loop of `RPartition` that allows `RPartition` to run in $O(n + m \log n)$ time. The algorithm is similar to the approach of [19] and [13]. Retrieving E' takes time proportional to the sum of degrees of X. The grouping operations on E' take $O(|E'|)$ time, by distributing the edges to buckets that are initially empty, and maintaining a list of nonempty buckets to allow retrieval of the groups without visiting empty buckets, leaving the buckets empty again. The selection of X ensures that each time a vertex is used as a pivot, the partition class that currently contains it is at most half as large as the class that contained it the last time it was used as a pivot. A vertex is therefore used $O(\log n)$ times as a pivot. A pivot on x requires $O(|N(x)|)$ time, giving the bound.

This clever idea is due to Spinrad, and initially appeared in an unpublished but influential manuscript, which he has circulated beginning in 1985 [26]. This paper was also the first to propose vertex partitioning as an algorithmic tool, and to recognize its importance to the modular decomposition and transitive orientation problems. Algorithm 6.27 is a variation of his time-bound argument that bounds the cost of partitioning inside all recursive calls to `RPartition`, instead of just in the main loop.

ALGORITHM 6.27. Vertex partitioning.

Partition $(G, \mathcal{P})$
  If $|\mathcal{P}| = 1$ return $\mathcal{P}$
  else
    Let X be a member of $\mathcal{P}$ that is not larger than all others
    Let E' be the edges of G in $X \times (V(G) - X)$.
    Group members of E' by endpoint in X
    Use each $x \in X$ as a pivot on all partition classes in $V - X$
    Regroup members of E' by endpoint in $V - X$
    Use each $v \in V - x$ as a pivot on all partition classes in X
    Let $\mathcal{Q}$ be the classes of $\mathcal{P}'$ now contained in X
    Let $\mathcal{Q}'$ be the classes of $\mathcal{P}'$ now contained in $V(G) - X$
    Return *Partition*$(G|X, \mathcal{Q}) \cup$ *Partition*$(G - X, \mathcal{Q}')$

Because of its simplicity, it would be the method of choice in a practical implementation of the algorithm of this paper, in our view. This is a bottleneck in the circular-arc graph recognition algorithm. To get a linear time bound for recognizing circular-arc

graphs we need a linear bound for `RPartition`. To do this we use a considerably more complicated strategy, which we give below. Before we do that, we show how `RPartition` can be used to obtain the Δ tree.

6.4.5. *Linear-time implementation.* From a theoretical standpoint, an $O(n + m \log n)$ bound for `RPartition` would be a bottleneck in our circular-arc graph recognition algorithm, and would result in an $O(n + m \log n)$ bound for it also.

To get a linear bound, we must make use of the considerably more difficult linear-time approach to vertex partitioning that is given in [18]. In this section we prove the following:

THEOREM 6.28. `RPartition` *can be carried out in* $O(n + m)$ *time.*

This is the most difficult part of the paper, since it requires a summary description of tricks that are explained in greater detail in [18]. There is no loss of continuity if one accepts Theorem 6.28 provisionally on a first reading and skips to Section 8.

The following are given in [18]:

THEOREM 6.29. *Let G be a prime graph, and let $\mathcal{P}$ be an arbitrary initial partition of V.* `Partition` *may be implemented so that it takes $O(n + m)$ time to halt with all partition classes singletons.*

THEOREM 6.30. *It takes linear time to find the modular decomposition of an undirected graph.*

By Theorem 6.29, it is immediate that we can run `Partition` in linear time on a prime graph; `RPartition` partitions V down to singleton sets. The paper solves the general transitive orientation problem through a reduction to prime graphs. In addition, the algorithm is specific to undirected graphs, and it is not immediate that it can be generalized even to prime interval matrices. For our application, we must be able to run `RPartition` on $U(T)$, where T is an interval matrix that need not be prime, and still get a linear time bound for `RPartition`.

We develop our algorithm in two steps. In the first we show how to perform `RPartition` on an undirected graph that need not be prime, in linear time, and in the second we show how to do it on $U(T)$, where $U(T)$ need not be prime.

6.4.6. *Running* `RPartition` *on a graph that need not be prime.* Let $G = (V, E)$ be an undirected graph, and let $\mathcal{M}_G$ denote the family of modules of G. Let T be a family of subsets of V where the transitive reduction of the containment relation is a tree, and where V and its singleton subsets are members of T. Moreover, suppose that each node of the tree is labeled *prime* or *degenerate*. Let $\mathcal{F}(T)$ denote the family of sets where $X \in \mathcal{F}(T)$ iff it is a node of T or a union of siblings whose parent is a degenerate node. T is an *M tree* on G if $\mathcal{M}_G \subseteq \mathcal{F}(T)$. If T is the modular decomposition of a graph, then it is an M tree, and $\mathcal{M}_G = \mathcal{F}(T)$.

In general, an M tree looks like the modular decomposition, except that when it is interpreted as the modular decomposition, it exaggerates the number of modules of G

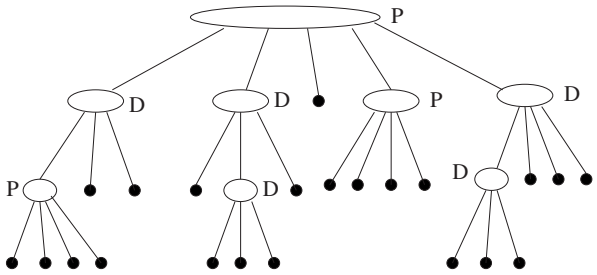


Fig. 9. An M tree is a tree defined on $G = (V, E)$. When interpreted as the modular decomposition, the family of sets that it claims are modules contains all modules, but also some nonmodules.

(Figure 9). However, it never implies that a module of G is not a module. If T_1 and T_2 are M trees, then T_2 is *more restrictive* than T_1 if $\mathcal{F}(T_2) \subset \mathcal{F}(T_1)$. The modular decomposition tree is the most restrictive one possible for G. The least restrictive M tree possible has V as a degenerate root and its singleton subsets as its children, since then $\mathcal{F}(T)$ is the power set of V.

The modular decomposition algorithm of [18] constructs a series of length $O(1)$ of M trees, each one more restrictive than the previous, until the modular decomposition is obtained. In constructing T_i from T_{i-1} in the sequence, the algorithm makes use of vertices that demonstrate that each member of $\mathcal{F}(T_{i-1}) - \mathcal{F}(T_i)$ are not modules. Each of these vertices is a nonmember of some of the sets that has both neighbors and non-neighbors in them, which discredits them as modules. The main insight behind the transitive orientation algorithm is that these same vertices can be used as pivots during vertex partitioning whenever a class $X \in \mathcal{P}$ arises that is a member of $\mathcal{F}(T_{i-1}) - \mathcal{F}(T_i)$. We call these vertices T_i's *pivots*.

Each node of an M tree is represented with an $O(1)$-sized structure that has a pointer to a doubly linked list of its children. The set that the node represents is given by its leaf descendants.

The members of $\mathcal{P}$ are implemented with doubly linked lists, and when splitting $Y \in \mathcal{P}$ with a pivot w, we remove neighbors of w from Y, and let what remains of Y stand for the remaining partition class. This avoids the cost of touching non-neighbors of w in Y. We call this a *removal operation.*

LEMMA 6.31 [18] (see Figure 9). *Given an M tree T on graph G, one may perform the following in time linear in the size of G:*

1. *Sort all adjacency lists so that the neighbors of each vertex x are given in order of their left-to-right appearance as leaves of T. For each node of the tree, the members of the node are therefore consecutive in the adjacency list of x.*
2. *Label each node U of the tree with a sorted list $SN(U)$ of vertices in $V - U$ that are adjacent to all members of U. $SN(U)$ is sorted in left-to-right appearance of its members as leaves of T, and the sum of lengths of these lists is linear in the size of G.*
3. *Label each node U that is not a module with the leftmost vertex $l(U) \in V - U$ and rightmost $r(U) \in V - U$ that is adjacent to some members of U and nonadjacent*

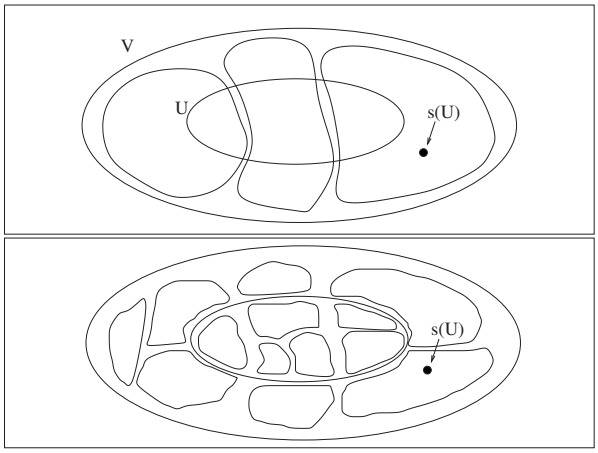


Fig. 10. A *restarting procedure* on an M tree performs pivots in G until every partition class is a node or a union of children of a degenerate node. The top illustration depicts a node U of the M tree that intersects more than one partition class in the partition of V. The bottom illustration depicts the situation after the restarting procedure for the tree is called. U now contains all partition classes that it intersects. At this point, U may be "processed," by performing a pivot on classes that U intersects, using the interval occupied by $N(s(U) \cap U)$ in $s(U)$'s adjacency list.

to others; *since U is not a module, at least one of these is defined. Let $s(U)$ denote one of them. U is also equipped with a pointer to the consecutive interval occupied by $N(s(U)) \cap U$ in $s(U)$'s adjacency list.*

The algorithm assigns a private copy of an adjacency-list representation of G to each T_i, and applies Lemma 6.31 to each. It uses a *restarting procedure* on each T_i, which initiates pivots on $\mathcal{P}$, halting only when $\mathcal{P} \subseteq \mathcal{F}(T_i)$. (See Figure 10.) To do this, it makes a call to the restarting procedure on T_{i-1}. When that restarting procedure halts, it makes use of T_i's pivots to split some classes, communicates these splits to T_{i-1} as capricious splits, and then calls the restarting procedure on T_{i-1} to get the partitioning restarted. The procedure fails to find any pivots to restart partitioning only if $\mathcal{P} \subseteq \mathcal{F}(T_i)$, in which case it is allowed to halt.

The restarting procedure is *linear* if it may be restarted $O(n + m)$ times without spending more than $O(n + m)$ total time, including the time to perform the pivots it generates and to maintain the list of unprocessed nodes. All of the restarting procedures in the sequence of M trees are linear ones.

Notice that a linear restarting procedure on the last tree in the sequence, $MD(G)$, would suffice for a linear-time implementation of `RPartition`. The restarting procedure could perform the `while` loop of `Partition`, halting when every partition class is a member of $\mathcal{F}(MD(G))$ (i.e., a module). A new call to `Partition(`$T|Z$, v`)` could then initiate the split $(\{v\}, Z)$ of a module Z and restart the restarting procedure again. By the definition of a linear restarting procedure, the time for all calls to the restarting procedure would be linear.

General linear restarting procedures are given in [18] for all trees in the sequence except for the very last one, which is the modular decomposition of G. This is unfortunate,

since doing this requires only a straightforward extension of the techniques used for the other restarting procedures, which must now be resurrected here. The reason that this was not done is that when the modular decomposition is available, transitive orientation of a comparability graph reduces to transitive orientation of prime graphs, and this step was unnecessary for transitive orientation.

The reason pivoting risks being wasteful is that the class that contains a pivot vertex x cannot be split by x, so time might be wasted traversing members of x's class in splitting up classes that do not contain x. After a sequence of later pivots, some of these might lie in a different class from x, and these portions of x's adjacency list might have to be retraversed to look for these elements. Bounding the number of retraversals is the main challenge.

The basic technique of [18] is to use the restarting procedure on T_{i-1} to impose an organized structure on the partition classes so that a set of classes that are to be split by a pivot x in T_i occupy a consecutive region of x's adjacency list, which is sorted by leaf order in T_i. This region of the list can then be discarded as it can have no effect on the results of future pivots. No elements in the list have to be retraversed later.

The last tree in the sequence before the modular decomposition is called an *M2 tree*. An M2 tree is an M tree with the following property: for any node U that is labeled degenerate, every union of children of U is a module of $G|U$ (but not necessarily of G).

To get the restarting procedure for the modular decomposition, we insert a new M tree $\mathrm{M_p}$ in the sequence between the M2 tree and the modular decomposition. ($\mathrm{M_p}$ is the "penultimate" tree.) We do this by creating a copy of the M2 tree, and performing the operations of Lemma 6.31 on it. For each degenerate node U, let $\mathcal{A}$ be its children that are modules of G. $\mathcal{A} = \{W : W \text{ is a child of } U \text{ such that } s(W) \text{ is undefined}\}$. We install a new child A of U representing $\bigcup \mathcal{A}$, and remove the members of $\mathcal{A}$ and their subtrees from the list of children of U and make them the list of children of A. We label A degenerate.

By concurrently traversing the lists $\{SN(A) : A \in \mathcal{A}\}$, we may either conclude that A is a module of G, or else find a first vertex x that is adjacent to a proper subset $\mathcal{A}_a$ of $\mathcal{A}$, and let $\mathcal{A}_n$ be the remaining subset that are nonadjacent to x. When x exists, we let $s(A) = x$, and create two new nodes A_a and A_n standing for $\bigcup \mathcal{A}_a$ and $\bigcup \mathcal{A}_n$. We remove the members of $\mathcal{A}_a$ from the doubly linked list of children of $\bigcup \mathcal{A}$ in $O(|\mathcal{A}_a|)$ time and make them children of A_a, and then move the remaining list of children of $\bigcup \mathcal{A}$ in $O(1)$ time and make it the list of children of A_n. We then recurse on $\mathcal{A}_a$ and $\mathcal{A}_n$, using what remains of the SN lists in each of these sets. The base case is reached when a recursive call is made on a single member of $\mathcal{A}$ or on a subset $\mathcal{A}_m$ of $\mathcal{A}$ such that $|\mathcal{A}_m| > 1$ and $\bigcup \mathcal{A}_m$ is a module in G. The latter case is recognized when what remains of the $SN()$ lists of members of $\mathcal{A}_m$ are exhausted without discovery of a splitting vertex x. In this case the corresponding node A_m is labeled degenerate, and its children are $\mathcal{A}_m$.

When we are done, we sort adjacency lists in $\mathrm{M_p}$'s private copy of G by leaf order in $\mathrm{M_p}$ so that each node of the tree occupies a consecutive portion of each adjacency list.

The restarting procedure given in [18] for the M2 tree is called *M2Resolve*(). As partitioning on $\mathcal{P}$ proceeds, *M2Resolve* a list of "unprocessed" nodes of T_i, each of which contains every partition class that intersects it. Though it is not necessary for the algorithm, it helps to understand it if one assumes that the list is ordered so that ancestors appear before descendants. The list supports permanent removal of a maximal

unprocessed node from the front of the list. (This node is used to assist in identifying promising pivots.) When a pivot is performed, it supports identification and insertion of all new nodes that now satisfy the requirement of containing partition classes that intersect them, in an appropriate order at the end of the list. The list can be regarded as an abstract data type that updates itself each time a partition class is split.

This requires an initial overhead of $O(n+m)$ time to set up. After that, the amortized time to update this list when a partition class Y is split by removing a set Y_a from Y is $O(|Y_a|)$, which is the cost of the removal operation. This is true whether the split is the result of a pivot, or is a capricious split of the class that is decided upon by some other process.

To initiate the pivoting, we call *M2Resolve*, beginning with the initial partition $\mathcal{P}$. When it returns, each member P in the refined partition $\mathcal{P}$ is either a node of the M2 tree or a union of children of a degenerate node of M2.

Let W be a maximal unprocessed node of M2. We remove W from the list. If the vertex $s(W)$ that splits W does not exist, W is a module, and we proceed to the next maximal unprocessed node in the list. The reason this is justified is that *M2Resolve* has halted. Each partition class that intersects W is either W itself, or contained in one of its children if W is prime, or is contained in a child or a union of children if W is degenerate. If W is a partition class, the class is a module, and no further work is required on it. A partition class that is contained in one of its children, C, will be dealt with when the unprocessed children of W are processed. If a partition class is a union of children, then since every union of children of W is a module of $G|W$ and W is a module of G, each such union is a module of G, and no further work is required on it.

If $s(W)$ exists, suppose first that W is prime in M2. We pivot on $s(W)$ by traversing only the portion of $s(W)$'s adjacency list that contains members of W. We find this portion using the pointer carried by W to $N(s(W) \cap W)$ in $s(W)$'s adjacency list. No members of the partition class containing $s(W)$ are encountered in this region of the list, since $s(W)$ is a vertex outside of W. Every partition class intersecting W is contained in W. These elements may now be discarded from W's adjacency list, as they are irrelevant to future pivots. These pivots ensure that if W is not a module, it now contains more than one partition class. W has now been processed, and we call *M2Resolve* again, which ensures that every partition class that intersects a child C of W is now contained in C. These will be dealt with when C is removed from the list of unprocessed nodes.

The procedure when W is degenerate is slightly more complicated, since W can have a child A in $\mathrm{M_p}$ that it does not have in the M2 tree. As in the prime case, we begin with a pivot on W with $s(W)$, discarding the members of W from $s(W)$'s adjacency list. Again let $\mathcal{A}$ be those children of W that are modules of G. For each child C not in $\mathcal{A}$, we pivot on the regions occupied by W in $s(C)$. Since C is a module of $G|W$, $s(C)$ lies outside of W. We may then discard members of W from $s(C)$'s adjacency list. A call to *M2Resolve* now returns with each partition class intersecting such a child C contained in C. By default, every partition class intersecting $A = \bigcup \mathcal{A}$ is also contained in A. Pivoting on $s(A)$ and removing members of A from $s(A)$'s adjacency list makes this true for its children A_a and A_n. We may do this recursively, halting when we reach the highest descendants of A that are modules. If such a descendant D contains more than one child of W in the M2 tree, then it is a module, and so is every union of its children

in the M_p tree. Calling *M2Resolve* now halts only when each partition class is either a union of children of D, in which case it is a module, or contained in a child C of D, in which case it will be dealt with when C is processed.

The procedure can only fail to generate effective pivots when every partition class is a module. That is precisely the halting criterion for a restarting procedure on $MD(G)$.

The total time preparing to restart *M2Resolve* is proportional to the number of elements discarded from adjacency lists, and is therefore linear. There are $O(n+m)$ calls to `M2Resolve()`, so the time these calls require is linear, since `M2Resolve()` is a linear restarting procedure. As explained above, the total amortized cost of maintaining the list of unprocessed nodes is linear. This gives the following:

LEMMA 6.32. *A linear restarting procedure may be constructed for the modular decomposition of an undirected graph.*

COROLLARY 6.33. `RPartition` *takes* $O(n+m)$ *time on an undirected graph.*

6.4.7. *A linear time bound for* `RPartition` *on an interval matrix.* If T is an interval matrix, then $X \subseteq V$ is a module of $U(T)$ iff it is a module of both G_n and G_{1c}. One problem is that we need to avoid working directly on these graphs, since their size is not linear in the size of $G(T)$. Fortunately, vertex partitioning on G_n can be simulated with vertex partitioning on G_{1c}, and vertex partitioning in G_{1n} can be simulated with vertex partitioning on G_c. The partition of a class induced by a pivot is identical in the simulation; only the interpretation of the classes, hence the order given to them in the linear order on $\mathcal{P}$, is changed. Adjusting the ordering of $\mathcal{P}$ to accommodate this reinterpretation takes $O(1)$ time per split, hence $O(n)$ time total.

Let $M_p(G_c)$ and $M_p(G_{1c})$ denote M_p trees for G_c and G_{1c}. We may use a restarting procedure for each of $MD(G_c)$ and $MD(G_{1c})$. Beginning with our initial partition, we call the restarting procedure for $MD(G_c)$, which refines $\mathcal{P}$ until every partition class is a module of G_c, using pivots in G_c.

It may be the case that not every partition class is a module of G_{1c}, so we communicate the list of removal operations that have been performed as capricious splits to $M_p(G_{1c})$, and initiate the restarting procedure for $MD(G_{1c})$. This performs pivots until all members of $\mathcal{P}$ are modules of G_{1c}. At this point, they may not be modules in G_c, so we may communicate the new removal operations back to $M_p(G_c)$ and return control to the restarting procedure for $MD(G_c)$.

Alternating between the restarting procedure of $MD(G_c)$ and $MD(G_{1c})$ continues refining $\mathcal{P}$, halting only when neither of the restarting procedures is able to refine $\mathcal{P}$ further. This occurs only when every member of $\mathcal{P}$ is a module of both G_c and G_{1c}, hence a module of $U(T)$. This is the halting criterion for a restarting procedure $MD(U(T))$.

Since each of the restarting procedures is restarted $O(n)$ times, they take $O(n + m(G_c))$ and $O(n + m(G_{1c}))$ time, respectively, for all pivots that they perform. This is linear in the size of $G(T)$. This gives the following:

LEMMA 6.34. `RPartition` *on* $U(T)$, *beginning with an arbitrary starting partition, takes* $O(n+m)$ *time.*

The following is now an immediate corollary:

THEOREM 6.35. *It takes $O(n+m)$ time to produce an interval realizer of an interval matrix.*

6.5. *The Case Where G Is Not a Circular-Arc Graph.* We now illustrate how halting conditions that are consistent with standard programming practice can be derived for the case when G is not a circular-arc graph. In this case Steps 1 or 2 may terminate early. Otherwise, they may output unreliable results. Therefore, we cannot assume that the precondition of Step 3 is met, which is that T is the intersection matrix of an interval graph. However, we show below that they halt in $O(n+m)$ time. Therefore, the representation of T with an edge labeling of G_{1c} has size $O(n+m)$.

Suppose T is not an interval matrix. If it has labels that identify some of the edges as double overlaps, which do not belong in an interval matrix, G can be rejected, as we have shown that this cannot happen when G is a circular-arc graph. Similarly, G can be rejected if the subgraphs G_1 and G_c given by the matrix are not symmetric (undirected).

Otherwise, the Δ modules and Δ relation are still defined on T, just as modules and Γ are defined on arbitrary graphs. `RPartition` produces linear extensions of alleged interval orientations which may not be interval orientations at all, just as the transitive orientation algorithm of [18] produces a nontransitive orientation when its input is not a comparability graph. When these alleged interval orientations are used to produce an interval realizer, this realizer must fail to realize T, since T is not an interval matrix. The analysis of the time bound of the algorithm is unaffected by conditions.

7. Details of Step 1: Creating an Intersection Matrix that Shares a Realizer with a Given Circular-Arc Graph. In this section we give the implementation of Step 1 of Algorithm 4.1. If G is a circular-arc graph, this step consists of finding an intersection matrix whose realizer is a realizer of G if G is a circular-arc graph. If G is not a circular-arc graph, then, by Lemma 5.1, the step is allowed either to halt because it detects that a precondition on the input that one can expect from circular-arc graphs is not met (thereby rejecting G), or to produce a meaningless output.

This section makes use of algorithmic results from a variety of sources, and is therefore not self-contained if one wishes to have a complete understanding of the implementation details.

This reduces to evaluating the conditions of Definition 4.2 at each adjacent pair of vertices. That is, for each adjacent pair $\{x, y\}$, we must evaluate $N[x] \subset N[y]$, $N[y] \subset N[x]$, and $N[x] \cup N[y] = V$ Then, at each adjacent pair $\{x, y\}$ such that $N[x] \cup N[y] = V$, we must check whether for each $z \in N[x] - N[y]$, $N[z] \subseteq N[x]$, and for each $z' \in N[y] - N[x]$, $N[z'] \subseteq N[y]$ in order to disambiguate Cases 4 and 5 of Definition 4.2.

7.1. *Evaluating $N[x] \subset N[y]$, $N[y] \subset N[x]$, and $N[x] \cup N[y] = V$ When G Is a Circular-Arc Graph.* The main insights of this section were developed by Eschen and Spinrad [9]. We improve their time bound to $O(n+m)$ for testing for each edge xy whether $N[x] \subset N[y]$, $N[y] \subset N[x]$, and $N[x] \cup N[y] = V$; they gave an $O(n^2)$ bound for performing this on all $\Theta(n^2)$ pairs of vertices.

We first show that we may assume by means of a reduction that G has no universal vertices or clique modules. We then select a vertex m_0 whose arc does not contain any other arc in some realizer of G, and such that $|N[m_0]| = O(m/n)$. For example, selecting m_0 to be a vertex of minimum degree satisfies these requirements.

Let $U = V - N[m_0]$. Since there are no clique modules, $N[x] \subseteq N[y]$ implies $N[x] \subset N[y]$. For testing for a neighborhood containment between x and y, we break the problem into two parts: testing whether $N[x] \cap U \subseteq N[y] \cap U$ and testing whether $N[x] \cap N[m_0] \subseteq N[y] \cap N[m_0]$. This test falls into different cases, depending on whether x and y are both in $N[m_0]$, both in U, or one is in $N[m_0]$ and the other is in U. These tests are made easy by the fact that $G|U$ is an interval graph and the edges of G that go between $N[m_0]$ and U have a special *chordal bipartite* structure.

LEMMA 7.1. *Let G be a circular-arc graph.*

1. *If y is a universal vertex, a circular-arc realizer for G can be obtained from any circular-arc realizer for $G|(V - \{y\})$, in $O(1)$ time.*
2. *If X is a clique module of G and $x \in X$, an intersection matrix for G can be obtained from any intersection matrix of $G|((V - X) \cup \{x\})$ in $O(|X|)$ time if X is given.*

PROOF. To obtain a realizer of G from a realizer of $G|(V - \{y\})$, insert $r(y)$ anywhere, and insert $l(y)$ immediately to the right of it. This causes it to cover the circle, except in an interval between $r(y)$ and $l(y)$ that has no endpoints of other arcs in it.

Let $X = \{x_1, x_2, x_3, \ldots, x_k\}$. A realizer of G can be obtained from a realizer of $G|((V - X) \cup \{x\})$, by replacing $l(x)$ with the sequence $(l(x_1), l(x_2), \ldots, l(x_k))$, and replacing $r(x)$ with the sequence $(r(x_1), r(x_2), \ldots, r(x_k))$. □

We apply the reduction Lemma 7.1 initially to G, eliminating all universal vertices and all but one vertex from each remaining maximal clique module, to obtain a graph G'. We find a realizer of G', and use Lemma 7.1 to find a realizer of G.

To show that this takes linear time, we must show a linear time bound for finding all maximal clique modules of a graph. For this, the following well-known lemma is useful.

LEMMA 7.2. *Given an adjacency-list representation of a graph G and a numbering of its vertices from 1 to n, it takes $O(n + m)$ time to put all adjacency lists into ascending order of vertex number.*

PROOF. Number the vertices from 1 to n, then perform a two-pass radix sort of the concatenation of the adjacency lists, with vertex of origin as the primary key and destination vertex as the secondary sort key. Each adjacency list is now consecutive and in sorted order. □

The origin of the following is unclear, but a proof can be found in [5], where the procedure is called *radix partitioning*.

LEMMA 7.3. *Let L be a list of nonempty strings of integers from $\{1, 2, \ldots, n\}$ whose combined length is m. It takes $O(n + m)$ time to partition L into maximal groups of identical strings.*

The process is similar to radix sorting, but achieves a better time bound because it is not required that the groups be given in any particular order.

COROLLARY 7.4. *It takes $O(n+m)$ time to find the maximal clique modules of a graph.*

PROOF. The key observation is that the clique modules are maximal sets of vertices with identical closed neighborhoods. Number the vertices from 1 to n. Sort all of the adjacency lists by Lemma 7.2. For each vertex v, add a copy of v to the spot where it belongs in the sort of its own adjacency list. This gives all closed neighborhoods, with the elements of each neighborhood listed in sorted order. Apply Lemma 7.3 to find the maximal groups of identical closed neighborhoods. □

We assume in the rest of Section 7 that G has no universal vertices or clique modules, hence no two vertices have identical closed neighborhoods.

THEOREM 7.5 [14]. *If G is a circular-arc graph with no clique modules or universal vertices, then $T(G)$ is the intersection matrix of a circular-arc realizer of G.*

Hsu calls a realizer of this matrix a *normalized model*, and works exclusively with normalized models. If G is a graph, performing a flip on a vertex in $T(G)$ yields a new matrix T' for a graph G', and T' gives the intersection types of a realizer of G'. It is not necessarily true that $T' = T(G')$, so a geometric flip may or may not give a normalized realizer. The sole significance to us of a normalized model is that it allows us to get an initial intersection matrix.

A *chordal bipartite graph* is a bipartite graph with no chordless cycles of length 6 or greater. Let n_1 and n_2 denote the sizes of its bipartition classes. A *neighborhood containment test* on two vertices x and y consists of evaluating the expression $N(x) \subseteq N(y)$. A *disjoint neighborhood test* consists of evaluating whether $N(x) \cap N(y)$ is empty.

THEOREM 7.6 [27]. *In a bipartite graph G with bipartition classes of sizes n_1 and n_2 and k pairs of vertices of G, it takes $O(n_1 n_2 + k)$ time either to determine that G is not chordal bipartite or else to perform neighborhood containment tests on the pairs.*

THEOREM 7.7 [8]. *Given a bipartite graph G with bipartition classes of sizes n_1 and n_2 and k pairs of vertices of G, it takes $O(n_1 n_2 + k)$ time either to determine that G is not chordal bipartite or else perform disjoint neighborhood tests on the pairs.*

THEOREM 7.8 [9]. *Let G be an arbitrary circular-arc graph, and let m_0 be a vertex whose arc contains no other in some circular-arc realizer of G. Let $D(N'[m_0], V, E_D)$ be a bipartite graph, where $N'[m_0]$ is a copy of $N[m_0]$, and $xy \in E_D$ iff $x \neq y$ and $xy \notin E$. Then D is chordal bipartite.*

In the statement of Theorem 7.8 in [9], m_0 is only required to be a vertex of minimum degree. However, this is only to ensure for the proof that m_0's arc contains no other arc

in some circular-arc realizer. The proof works without modification for this version of the theorem.

THEOREM 7.9. *Let G be a graph and let $U \subseteq V$. It takes $O(n+m+k)$ time either to determine that $G|U$ is not an interval graph or to evaluate $N[x_i] \cap U \subseteq N[y_i] \cap U$ and $(N[x_i] \cap U) \cup (N[y_i] \cap U) = U$ at k pairs $\{x_i, y_i\}$ of vertices such that $y_i \in U$.*

PROOF. It takes $O(n+m)$ time to determine that $G|U$ is not an interval graph or to produce an interval realizer of $G|U$ [3]. If it is an interval graph, create a list L of the left endpoints in the realizer in the order in which they appear, and a list R of the right endpoints in the order in which they appear. For each vertex w of G, compute the leftmost right endpoint $L(w)$ and the rightmost left endpoint $R(w)$ among neighbors of w in the realizer of $G|U$. Compute also the leftmost right endpoint $L'(w)$ and the rightmost left endpoint $R'(w)$ among non-neighbors of w in the realizer of $G|U$. Computing $L(w)$ and $R(w)$ clearly takes $O(|N(w)|)$ time. To compute $L'(w)$, mark neighbors of w in the realizer, in $O(|N(w)|)$ time. Then traverse R from left to right until a right endpoint of an unmarked vertex is encountered; this is $L'(w)$. $R'(w)$ can be found by a symmetric operation on L. All but one endpoint encountered in each traversal is a neighbor of w, so this takes $O(|N(w)|)$ time. Then unmark neighbors in $O(|N(w)|)$ time to reinitialize the realizer. The entire operation takes $O(|N(w)|)$ time, hence $O(n+m)$ time over all vertices.

For $\{x_i, y_i\}$ such that $y_i \in U$, $N[x_i] \cap U \subseteq N[y_i] \cap U$ iff $l(y_i) < L(x_i)$ and $r(y_i) > R(x_i)$. $(N[y_i] \cap U) \cup (N[x_i] \cap U) = U$ iff $l(y_i) < L'(x_i)$ and $r(y_i) > R'(x_i)$. □

THEOREM 7.10 [9]. *Let G be a graph and let $U \subseteq V$. Then it takes $O(n+m+n|V-U|)$ time either to determine that $G|U$ is not an interval graph or else to evaluate $N[x_i] \cap U \subseteq N[y_i] \cap U$ at all pairs $\{x_i, y_i\}$ of vertices such that $x_i \in U$ and $y_i \in V - U$.*

Nonadjacent pairs always have the same intersection type, so we use a sparse representation of the intersection matrix that consists of labeling the edges of G with their intersection type. The following two theorems now suffice to compute $T(G)$ in linear time.

THEOREM 7.11. *It takes $O(n+m)$ time either to determine that a graph G is not a circular-arc graph or else to evaluate the following boolean expressions at each edge xy in G:*

1. $N[x] \subseteq N[y]$;
2. $N[y] \subseteq N[x]$;
3. $N[x] \cup N[y] = V$.

In their conference paper, Eschen and Spinrad give an $O(n^2)$ bound, but omit the proof of part 2 due to space considerations. However, it is a variant of the proof of part 1. They do not give a tighter bound, presumably because this suffices to get their $O(n^2)$ bound for circular-arc graph recognition. However, it is not hard to get an $O(n+m)$

bound using their methods, if one is interested in evaluating these expressions only at adjacent pairs of vertices. For completeness, we give the details here.

PROOF. A vertex of minimum degree satisfies the requirements of m_0 in Theorem 7.8, and has degree $O(m/n)$ in G. The size of the bipartite adjacency matrix for the graph D given by the theorem is $O((m/n)n) = O(m)$. Let U denote $V - N[m_0]$.

Suppose G is a circular-arc graph. Since G is a circular-arc graph, removal of $N[m_0]$ uncovers part of the circle in any circular-arc realizer of G. Thus, $G|(V - N[m_0]) = G|U$ is an interval graph. It takes $O(n + m)$ time to find an interval model of $G|U$ [3].

1. Consider pairs of the form $\{x, y\} \subseteq N[m_0]$. By Theorem 7.6, we may evaluate $N[\bar{G}, x] \subseteq N[\bar{G}, y]$, which is true iff $N[y] \subseteq N[x]$. Using Theorem 7.7, we test whether the neighborhoods of x and y are disjoint in D, which tells whether $N[x] \cup N[y] = V$ in G.
2. Next, consider adjacent pairs of the form $\{x, y\} \subseteq U$. It takes linear time for all such pairs to evaluate $N[x] \cap U \subseteq N[y] \cap U$, by Theorem 7.9, and linear time to evaluate $N(D, y) \subseteq N(D, x)$ in D, by Theorem 7.6, which gives $N[x] \cap N[m_0] \subseteq N[y] \cap N[m_0]$. The results of these two tests give $N[x] \subseteq N[y]$. To evaluate $N[x] \cup N[y] = V$, $(N[x] \cap N[m_0]) \cup (N[y] \cap N[m_0]) = N[m_0]$ iff $N(D, x) \cap N(D, y)$ is empty. By Theorem 7.7, we can compute this for all these adjacent pairs in linear time. We evaluate $(N[x] \cap U) \cup (N[y] \cap U) = U$ using Theorem 7.9.
3. Pairs of the form $\{x, y\}$ with $x \in N[m_0]$ and $y \in U$ are handled as in part 2, except that $N[x] \subseteq N[y]$ must be obtained with Theorem 7.10. Since $|N[m_0]|$ is $O(m/n)$, the time bound is still $O(m)$.

Now, suppose that G is not a circular-arc graph. If Theorems 7.6, 7.7, 7.9, or 7.10 fail to deliver the required results, then this can only be because they determined that a subgraph failed to satisfy a property that we have shown that it must satisfy if G is a circular-arc graph. □

By itself, Theorem 7.11 does not give circular-arc graph recognition, because it allows for the possibility that when G is not a circular-arc graph, G fails to be rejected, and instead the claimed tests are performed correctly.

7.2. *Disambiguating Cases* 4 *and* 5 *of Definition* 4.2. In this subsection we show the following, which is all that remains to do in order to disambiguate Cases 4 and 5 of Definition 4.2:

THEOREM 7.12. *It takes $O(n + m)$ time either to determine that G is not a circular-arc graph or else to evaluate the following boolean expressions at each edge xy in G such that $N[x] \cup N[y] = V$:*

For each $z \in N[x] - N[y]$, $N[z] \subseteq N[x]$, and for each $z' \in N[y] - N[x]$, $N[z'] \subseteq N[y]$.

An $O(n^2)$ bound for evaluating this in the special case where the vertices can be partitioned into two cliques is given in [9]. We need a linear bound for arbitrary circular-arc graphs.

The strategy of the proof can be summarized as follows. If $N[x] \cup N[y] = V$, then we can reduce the problem to one we have already solved, by flipping y in some realizer to obtain y' and testing whether $N[y'] \subseteq N[x]$ in the graph represented by the new realizer. We do not yet know a realizer of G, but Theorem 7.11 provides enough information to allow us to deduce the effect on G of flipping y.

However, to obtain the linear time bound, we must batch these tests together by adding to G a copy y' of one vertex y out of every pair $\{x, y\}$ such that $N[x] \cup N[y] = V$. Let V be the original vertices and let F be the copies. We then flip all the copies to obtain a graph H with vertex set $V \cup F$. We then perform the comparison on x and y by testing whether $N[y'] \cap V \subseteq N[x] \cap V$ in H.

This gives rise to an obstacle in applying the technique: If $x', y' \in F$, then we do not know whether they are adjacent in H. This depends on whether xy falls into Case 4 or Case 5. If xy is a single overlap in G, then $x'y'$ is a single overlap in H, which is an adjacency, but if xy is a double overlap, then $x'y'$ fails to be an edge in H. Thus, we do not know $H|F$, and we must avoid use of its edges when applying our tests.

Let m_0 be a vertex whose arc contains no other arc in a circular-arc representation of H. We use two key facts in order to employ techniques from the previous subsection:

1. $H|(V - N[m_0]) = G|(V - N[m_0])$ is an interval graph.
2. Let D be the bipartite graph $D(N[m_0], F \cup V)$ that is obtained as described by Theorem 7.8 applied to H. Then D is chordal bipartite, as is every induced subgraph of D.

We split the test into two parts: testing whether $N[y'] \cap (V \cap N[m_0]) \subseteq N[x] \cap (V \cap N[m_0])$ and testing whether $N[y'] \cap (V - N[m_0]) \subseteq N[x] \cap (V - N[m_0])$. These tests are performed using the two key facts and techniques similar to those of Section 7.1. How they are applied depends on which of x and y' are neighbors of m_0.

Computing D requires knowledge of edges of $H|F$, which we do not have. We get around this problem by making use of two induced subgraphs D_1 and D_2 of D, neither of which requires information about $H|F$ to compute, and which suffice to make the required tests of neighborhood containments. This trick resolves all of the cases, except the one where x is a neighbor of m_0 in H and y' is not. In this case we must improvise a new technique, which is based on a type of graph called a *probe interval graph*, which we define below.

A remaining issue is that, to apply our techniques, we must identify a vertex m_0 of H such that there is a circular-arc realizer of H where arc m_0 contains no other arc, $|N[m_0]| = O(m/n)$, and $m_0 \notin F$. The requirement that m_0 contain no other arc is necessary to apply Eschen and Spinrad's theorems. The requirement that $|N[m_0]| = O(m/n)$ is required for the time bound. That $m_0 \notin F$ is required because we need to know $N[m_0]$, and if $m_0 \in F$, this requires knowledge of $H|F$.

We first show how to find an m_0 that satisfies all of the requirements with the possible exception of $m_0 \notin F$. If $m_0 \in F$, we modify H, V, and F slightly so that it still suits our purposes and $m_0 \notin F$.

This concludes the summary of the strategy. We now give the detailed proof.

A *probe interval graph* is a graph $G = (V, E)$ and a partition $\{P, N\}$ of V such that $G|P$ is an interval graph, N is an independent set, and there is an interval realizer on V where edges of $P \times V$ can be realized with an interval model. There are no restrictions

on adjacencies and nonadjacencies assigned by the realizer among members of N. Such a realizer of G is a *probe interval realizer*.

THEOREM 7.13 [20]. *Given a graph G and a partition $\{P, N\}$ of its vertices, it takes $O(n + m + n|N|)$ time to construct a probe interval model or else determine that G is not a probe interval graph with respect to the partition $\{P, N\}$.*

Johnson and Spinrad showed an $O(n^2)$ variant of Theorem 7.13 [15]. We have modified it to get an $O(n + m \log n)$ bound [21], and the proof of Theorem 7.13 differs from this proof by allowing $O(n)$ time, rather than $O(\log n)$ time, to incorporate each nonprobe into the model.

THEOREM 7.14. *Let G be a graph, let $U \subseteq V$, and let $\{P, N\}$ be the partition of U. It takes $O(n + m + n|V - P|)$ time either to determine that $G|U$ is not a probe interval graph with respect to the partition $\{P, N\}$, or else evaluate $N[x] \cap P \subseteq N[y] \cap P$ at all adjacent pairs $\{x, y\}$ such that at most one of x and y is in $V - U$.*

PROOF. If $G|U$ is not a probe interval graph with respect to $\{P, N\}$, then we may detect this in linear time, by Theorem 7.13. Otherwise, we obtain a probe interval model R that realizes $G|U$. The algorithms of Theorems 7.9 and 7.10 can be trivially adapted to the problem. For Theorem 7.9, compute $L(w)$, $R(w)$, $L'(w)$, and $R'(w)$ in $R|P$, and let $l(y)$ and $r(y)$ denote the position of $y \in U$ relative endpoints of P. The test is now identical. An identical adaptation suffices for the algorithm of Theorem 7.10 given in [9]. □

ALGORITHM 7.15. Build an auxilliary graph for Theorem 7.12.

- Compute neighborhood containments in G and pairs of the form $xy \in G(T)$ and $N[x] \cup N[y] = V$, using Theorem 7.11.
- For each $xy \in G(T)$ such that $N[x] \cup N[y] = V$:
  - — Find one of the two vertices, say, x, with degree at least as large as the other.
  - — Insert a copy x' into G if one has not already been inserted, and make it adjacent to x, thereby making $\{x, x'\}$ a clique module.
- We call this graph H', and the set of new vertices F.

  At this point, let us pause to define some objects that we do not compute explicitly, but that we can now compute limited information about. Let R' any the realizer of H' derived from a realizer of $T(G)$ by letting each added arc $x' \in F$ properly contain its copy x in V. That is x and x' have identical arcs in R', except that the arc corresponding to x' has been stretched slightly so that it properly contains that of x. (We do this to avoid the complications of realizers where endpoints of arcs coincide.) Let H be the circular-arc graph that results from flipping each $x' \in F$ in R'.
- Find a vertex m_0 that minimizes $|N[H, m_0] \cap V|$.
- Return m_0, $H|V$, and the edges of H that go between V and F.

LEMMA 7.16. *Algorithm* 7.15 *takes $O(n + m)$ time and $|F| = O(m/n)$.*

PROOF. The size of H' is $O(n+m)$. Since there are $O(m)$ edges in H', the sum of degrees of members of F in H' is $O(m)$ in H'. Each vertex in F has degree $\Omega(n)$ in H', so $|F| = O(m/n)$. We may spend $O(n)$ time on each member of F without exceeding $O(n+m)$. Since each member of F has $\Omega(n)$ incident edges in H' and $O(n)$ incident edges in H, the size of H is also $O(n+m)$.

The critical observation is that the algorithm does not need to compute edges in $H|F$ in order to be able to return the result. $H|V = G$, so these edges are already available. For $x \in V$ and $y' \in F$, let y be the unflipped copy of y' in V. Since G has no clique modules, xy' fails to be an edge of H iff $N[G,x] \subset N[G,y]$. $N[G,x] \subset N[G,y]$ only if x and y are adjacent in G, so the results of this test are known from the first step of the algorithm. Since $|F|$ is $O(m/n)$, the time bound follows for these pairs. □

LEMMA 7.17. *Let m_0 be a vertex of H that minimizes $|N[H,x] \cap V|$. If $m_0 \in V$, then there is a realizer R of H such that arc m_0 contains no other.*

PROOF. Recall that G has no clique modules or universal vertices. Let R_G be the unknown realizer of $T(G)$ given by Theorem 7.5. Let R' be the realizer of H' obtained from R_G by adding a copy y' of each arc y of R that has a duplicate element of F in H'. To disambiguate what happens to the relationship of y and y' when y' is flipped, we assume that the endpoints of y' have been stretched slightly so that y' properly contains y. In the circular ordering of endpoints in R', the endpoints of y and y' form two consecutive pairs. Let R_H be the realizer of H obtained by flipping the members of F.

Let z' be an arc such that $N[H,z'] \cap V = N[H,m_0] \cap V$. Since m_0 minimizes $N[H,x] \cap V$, only arcs of this form could be be contained in arc m_0 in R_H. Thus, if z' does not exist, the claim holds. Otherwise, if $z' \in V$, then m_0 and z' are a clique module of G, a contradiction. Thus, z' is the flipped copy of some $z \in V$. Because of the way the copy of z' is created, z and z' are nonadjacent, hence so are m_0 and z. The endpoints of z and z' make two adjacent pairs in the circular ordering of endpoints in R_H, so if arc z' fails to contain arc m_0 in R_H, z is a neighbor of m_0 in G, a contradiction. □

In what follows, we let $N[H,x]$ denote the neighborhood in H of $x \in V \cup F$ in H, and let $N[x]$ denote the neighborhood in G of $x \in V$.

Let m_0 be the vertex returned by Algorithm 7.15. We need m_0 to be a member of V. Let D be the chordal bipartite graph on H given by Theorem 7.8. Let $D_1(X \cup Y, V)$ denote the subgraph induced by V and the copies of $X \cup Y$ in D, and let $D_2(X, V \cup F)$ be the subgraph induced by the copies of $V \cup F$ and the copies of X. Neither of these graphs requires knowledge of $H|F$ to compute. They are both chordal bipartite because they are induced subgraphs of D.

If $m_0 \in F$, m_0 is the twin of some $x \in V$, and each $y' \in F$ is the twin of some $y \in V$. If x and y are adjacent and $N[x] \cup N[y] = V$, then we do not know whether m_0 and y' are adjacent, since we do not know whether x and y have a single or double overlap. However, since we already know neighborhood containments among adjacent pairs in V we can find out in $O(n)$ time for all $z \in N[x] - N[y]$, whether $N[z] \subseteq N[x]$. Similarly, we can find for all $z' \in N[y] - N[x]$ whether $N[z] \subset N[y]$. (As we see below, however, this second test is unnecessary.) Since $|F| = O(m/n)$, all of these evaluations take

$O(m)$ time. This disambiguates Cases 4 and 5 on edges incident to x in G, which gives all edges of $H|(V \cup \{m_0\})$ and all edges of H between m_0 and members of $F - \{m_0\}$. At this point, we reset $G = H|(V \cup m_0)$, $V = V \cup \{m_0\}$, $F = F - \{m_0\}$. We remove universal vertices and collapse clique modules as before. We rerun Algorithm 7.15, but this time, there is no need to insert a copy of m_0 and flip it, since G already has one. We select the same m_0 to return as the first time. It is still true that $|N[H, m_0] \cap V|$ is minimized. Now, $m_0 \in V$, and Algorithm 7.15 gives us $H|V$ and edges of H from V to F. By Lemma 7.17, m_0's arc contains no other arc in some realizer of H.

For each pair x, y such that $N[x] \cup N[y] = V$, at least one of x and y has a copy in F. Suppose without loss of generality that y has a copy y' in F. Suppose $N[x] \cup N[y] = V$. If for some $z \in N[x] - N[y]$, $N[z] \not\subset N[x]$, then z has a neighbor that x does not have, and this neighbor must be a neighbor z' of y that x does not have. Since $N[x] \cup N[y] = V$, this implies that $z' \in N[y] - N[x]$, and $N[z'] \not\subset N[y]$, since z' has a neighbor, z, that y does not have. Test for all $z \in N[x] - N[y]$ whether $N[z] \subset N[x]$ suffices for the seemingly stronger requirement of the fourth condition in the definition of $T(G)$, where the condition on z' is redundant. However, the non-neighbors of y in V are exactly the neighbors of y' in V whose neighborhoods in V are contained in y'. *The condition is true iff* $N[H, y'] \cap V \subseteq N[H, x] \cap V$.

Thus, for Theorem 7.12, it remains to find for each $\{x, y'\}$ with $x \in V$ and $y' \in F$ whether $N[H, y'] \cap V \subseteq N[H, x] \cap V$.

ALGORITHM 7.18. Evaluate $N[H, y'] \cap V \subseteq N[H, x] \cap V$ for each adjacent pair xy' in H with $x \in V$ and $y' \in F$.

Let $X = N[H, m_0] \cap V$ and $Y = N[H, m_0] \cap F$. Note that $H|(V - X) = G|(V - N[m_0])$ is an interval graph.

Break the problem into two parts: $N[H, y'] \cap X \subseteq N[H, x] \cap X$ and $N[H, y] \cap (V - X) \subseteq N[H, x] \cap (V - X)$.

Case 1: $x \in V - X$, $y' \in Y$.

(a) $N[H, y'] \cap (V - X) \subseteq N[H, x] \cap (V - X)$. Apply the algorithm of Theorem 7.10.
(b) $N[H, y'] \cap X \subseteq N[H, x] \cap X$. Apply the algorithm of Theorem 7.6 to D_2.

Case 2: $x \in V - X$, $y' \in F - Y$.

(a) $N[H, y'] \cap (V - X) \subseteq N[H, x] \cap (V - X)$. Apply the algorithm of Theorem 7.10.
(b) $N[H, y'] \cap X \subseteq N[H, x] \cap X$. Apply the algorithm of Theorem 7.6 to D_2.

Case 3: $x \in X$, $y' \in Y$. $N[H, y'] \cap V \subseteq N[H, x] \cap V$. Apply the algorithm of Theorem 7.6 to D_1.

Case 4: $x \in X$, $y' \in F - Y$.

(a) $N[H, y'] \cap X \subseteq N[H, x] \cap X$. Apply the algorithm of Theorem 7.6 to D_2.
(b) $N[H, y'] \cap (V - X) \subseteq N[H, x] \cap (V - X)$. $H_I = H|((V - X) \cup (F - Y))$ is an interval graph, since a realizer of H fails to cover m_0. Let H_P be the probe interval graph obtained by omitting edges of $H|(F - Y)$ from H_I. Build a probe

interval model for H_P using the algorithm of Theorem 7.13. Apply the algorithm of Theorem 7.14 on $H|(V \cup (F - Y))$, using $U = (V - X) \cup (F - Y)$.

For the time bound, the matrices D_1 and D_2 have $O(m)$ entries each, since $|X \cup Y|$ is $O(m/n)$. Theorem 7.6 gives a linear bound for the cost of handling pairs in Cases 1(b), 2(b), 3, and 4(a). $|Y|$ and $|F - Y|$ are each $O(m/n)$, so Theorem 7.10 gives a linear bound for Cases 1(a) and 2(a), Theorem 7.13 gives a linear bound for building the probe model for Case 4(b), and Theorem 7.14 gives a linear bound for performing the neighborhood containment tests required for Case 4b, or else one of these theorems fails to deliver the required tests because a subgraph fails to have a property that it must have if G is a circular-arc graph. In any case, the algorithm halts in linear time.

This completes the proof of Theorem 7.12.

8. Details of Step 2: Finding a Set of Algebraic Flips to Turn an Intersection Matrix into an Interval Matrix. In this section we give the implementation of Step 2 of Algorithm 4.1.

We proceed by incrementally performing flips to transform an intersection matrix for the original input graph into an interval matrix. At each point, we let T denote the current state of the matrix, and let G_c, G_n, G_1, and G_2 refer to those graphs in T. The matrix passes through the following stages:

T_0: T_0 is an intersection matrix of the graph given as the input to the recognition problem.

T_1: T_1 is a matrix that has a vertex v_0 of degree $O(m/n)$, and all members of $N(G(T_1), v_0)$ have a single overlap relation with v_0.

For notational convenience, let $U = N(G(T_1), v_0)$ and $W = \bar{N}(G(T_1), v_0)$. (It will also be the case that $U = N(G(T_2), v_0) = N(G(T_3), v_0)$, and $W = \bar{N}(G(T_2), v_0) = \bar{N}(G(T_3), v_0)$.) Let $\mathcal{P}_W = \{D_1, D_2, \ldots, D_k\}$ be the components of $G_{1n}|W$. All edges between these are edges of G_c. We show that we may assume that all directed edges of D_c between these components are directed from left to right in this list. Let $A(D_1)$ denote the smallest interval on the circle that is disjoint from v_0 and contains all members of D_1. ($A(D_1)$ can be thought of as the region of the circle occupied by D_1.) For $2 \leq i \leq k$, all arcs in D_i contain $A(D_1)$.

T_2: (see Figure 11) T_2 shares the above properties with T_1, but $G_n|U$ and $G_2|U$ are empty, and all vertices in each component of $G_c|U$ cover a single endpoint of v_0 in any circular-arc realizer of T_2. Because of this, $xy \in G_2$ implies $x, y \in U$, so G_2 is empty.

Let $\mathcal{Q} = \{C_1, C_2, \ldots, C_h\}$ denote the components of $G_c|U$ that contain an arc with an endpoint in $A(D_1)$. If two members of $\mathcal{Q}$ cover opposite endpoints of v_0, flipping all members of one of them will make them cover the same endpoint.

T_3: (see Figure 12) The components of $\mathcal{Q}$ are flipped, as necessary, so that they all cover the same endpoint of v_0, and they all, therefore, fail to cover one endpoint, p, of $A(D_1)$. T_3 is the resulting intersection matrix. All arcs in $V - \bigcup \mathcal{Q}$ either contain $A(D_1)$ or are disjoint from it. These can be distinguished from each other by their relationship to any member of D_1 in T_3.

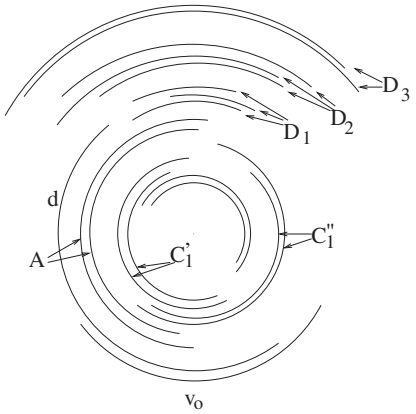


Fig. 11. The set W of non-neighbors of v_0 can be divided into components $\{D_1, D_2, \ldots, D_k\}$ of G_{1n}, which are ordered by containment. In T_2 the set U of neighbors of v_0 is a clique and all pairs in U are adjacent in G_c or G_1. Each component $G_c|U$ covers one endpoint of v_0's arc. $\mathcal{Q}$ is the set of components of $G_c|U$ that have a member with an endpoint inside an arc in D_1. In this example, $\mathcal{Q} = \{C_1', C_1'', A\}$. C_1' exemplifies $\mathcal{Q}'$, the set of components that have a member with different relationships to members of D_1, and C_1'' exemplifies $\mathcal{Q}''$, the set of components that are not members of $\mathcal{Q}'$, but whose members do not all have the same relationship to D_1. A exemplifies $\mathcal{A}$, the members of $\mathcal{Q}$ that have an overlap relationship with all members of D_1. Arc d is not in a set in $\mathcal{Q}$ because it has no endpoint inside a member of D_1.

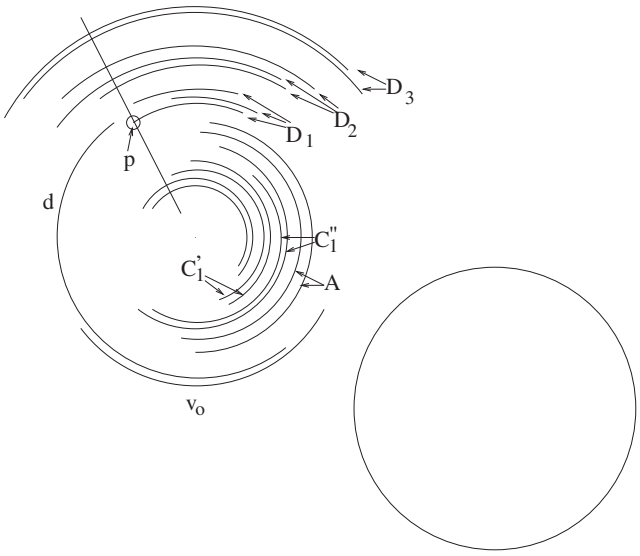


Fig. 12. Flipping the members of $\mathcal{Q}$, as needed, so that they cover the right endpoint of v_0, yields T_3. At this point, one endpoint p of the area occupied by D_1 is contained in those arcs that contain all members of D_1 in T_3, and only in those arcs. Flipping these yields T_4, leaving a region of the circle adjacent to p uncovered. T_4 is therefore an interval matrix.

T_4: Flipping the arcs in T_3 that contain $A(D_1)$ leaves the region of the circle adjacent to p and outside of $A(D_1)$ uncovered by any arc. T_4 is therefore an interval matrix.

8.1. *Turning T_0 into T_1.* To find the flips we need to perform in T_0 to obtain an interval matrix, we select a vertex v of minimum degree in G. When vertices are tied for minimum degree, we select v to be one whose arc contains no other arcs. An arc cannot contain another that has higher degree, so v always exists. If v has no double-overlap relationships, we select $v_0 = v$. Otherwise, we select a double-overlap neighbor of v that is contained in no other arc, flip it, and set v_0 to be the corresponding vertex. In either case, v_0 now has no double-overlap relations, and has at most the degree of v, which is $O(m/n)$. We then flip every arc that contains v_0. Now all arcs that intersect v_0 have a single-overlap relationship with it. T_1 is the state of T at this point.

LEMMA 8.1. *Every edge incident to v_0 in $G(T_1)$ is a single-overlap relationship in a realizer of T_1, and $|U|$ is $O(m/n)$.*

8.2. *Turning T_1 into T_2.* If for a vertex $x \in U$, x contains v_0's right endpoint, then flipping x will make it contain v_0's left endpoint instead.

If $x, y \in U$ and x and y contain the same endpoint, then they intersect, and since they fail to contain the other endpoint, they do not cover the circle, and they do not have a double intersection. Therefore, if xy is an edge of G_2 or G_n in T_1, then x and y contain opposite endpoints of v_0. $G_{2n}|U$ is bipartite, and the arcs of two vertices in opposite bipartition classes of a component of G_{2n} must contain opposite endpoints. If arc x contains arc y, then they contain the same endpoint of v_0, since they each miss an endpoint, so vertices in the same component of D_c contain the same endpoint of v_0. These observations give a unique bipartition of each component of $(D_c \cup G_{2n})|U$ into sets of vertices containing opposite endpoints. Flipping one of these sets makes them contain the same endpoint, eliminating all edges of G_n and G_2 from the component. Edges of D_c internal to a flipped set are flipped once at each endpoint, and remain edges of D_c, but with their orientations reversed. Edges leading out of the component are edges of G_1 and are unaffected by flipping. This flipping therefore turns the component of $D_c \cup G_{2n}$ into a component of D_c. Doing this for each component of $(D_c \cup G_{2n})|U$ makes $G_n|U$ and $G_2|U$ empty.

LEMMA 8.2. *Lemma* 8.1 *applies to T_2. In addition, all arcs in each of the components of $G_c|U$ contain the same endpoint of v_0 in any realizer of T_2.*

8.3. *Turning T_2 into T_3.* (See Figure 11.) Let $\mathcal{Q}' = \{C'_1, C'_2, \ldots, C'_{h'}\}$ be the components of $D_c|U$ that have a member that distinguishes members of D_1, and let $\mathcal{Q}'' = \{C''_1, C''_2, \ldots, C''_{h''}\}$ be the set $\{C : C$ is a component of $D_c|U$, $C \notin \mathcal{Q}'$, and there is a member of D_1 that distinguishes members of $C\}$. Let $\mathcal{A}$ denote the set set $\{A : A$ is a component of $G_c|U$ such that $A \times D_1 \subseteq G_1\}$. $\mathcal{Q} = \mathcal{Q}' \cup \mathcal{Q}'' \cup \mathcal{A}$.

We obtain T_3 from T_2 as follows:

Case A. If $\mathcal{Q}''$ is nonempty, we use a single rule to decide which members of $\mathcal{Q} = \mathcal{Q}'' \cup \mathcal{Q}'$ to flip to get them all to cover a single endpoint of v_0.

Case B. If $\mathcal{Q}''$ is empty, we use a different rule to decide which members of $\mathcal{Q}'$ to flip to get them all to cover a single endpoint of v_0.

Let T_3' denote the resulting matrix. We show that there is a realizer of T_3' where all members of $\mathcal{A}$ also cover this same endpoint, so $T_3 = T_3'$.

8.3.1. *Case A: $\mathcal{Q}''$ is nonempty.* We say that if $xz \in G_n$ and $yz \in G_1$, then y has a *stronger* relationship to z than x does, since y and z intersect, but x and z do not. If $xz \in G_1$ and $yz \in G_c$, then y again has a stronger relationship with z than x does, since it is a full containment, rather than a partial overlap. Similarly, if $xz \in G_n$ and $yz \in G_c$, then the relationship of y to z is stronger than the relationship of x to z.

For $x, y \in U$, let $x \preceq y$ denote that for every $z \in W$, the relationship of y to z is at least as strong as the relationship of x to z. That is, $x \preceq y$ if $N(G_{1c}, x) \cap W \subseteq N(G_{1c}, y) \cap W$ and $N(G_c, x) \cap W \subseteq N(G_c, y) \cap W$. If A and B are two components of $G_c|U$, then let $A \preceq B$ denote that for every $x \in A$ and $y \in B$, $x \preceq y$. Let $A \sim B$ denote that $A \preceq B$ or $B \preceq A$.

LEMMA 8.3. *If A and B are components of $G_c|U$ and cover the same endpoint of v_0 in a circular-arc realizer of T_2, then $A \sim B$.*

PROOF. Suppose A and B both cover the right endpoint of v_0 in some circular-arc realizer R. Let $A(W)$ denote the smallest interval of the circle that is disjoint from v_0 and contains every arc in W. (Think of $A(W)$ as the "region occupied by W.") Then for $x \in A$ and $y \in B$, x and y both extend counterclockwise into $A(W)$. If $A \preceq B$ and $B \preceq A$, the lemma holds. Suppose without loss of generality that y extends farther than x does, so that $y \not\preceq x$. Then every arc in W that is contained in x is also contained in y, and every arc in W intersected by x is also intersected by y. Therefore, $x \preceq y$.

Since A and B contain no arc that covers the left endpoint of v_0, $R' = R|(\{v_0\} \cup A \cup B)$ is an interval realizer of $T' = T|(\{v_0\} \cup A \cup B)$. The order of right endpoints in this realizer is a linear extension of an interval orientation of T'. The root node of the modular decomposition of T' is a degenerate node whose children, $\{v_0\}$, A, and B, are components in the complement of G_1. G_{1n} is transitively oriented in the interval orientation, so all edges of G_1 between A and B must either be directed from A to B, or from B to A by Lemma 6.1. Since $x \preceq y$ and $y \not\preceq x$, xy is oriented from x to y. Each arc of B extends at least as far to the right as every arc of A, hence at least as far into the region occupied by W in R. It follows that $A \preceq B$. □

G_2 is empty, D_c is transitive, and $\{D_1, D_2, \ldots, D_k\}$ are the children of the root of the modular decomposition of $G_c|W$. By Lemma 6.1, we may assume without loss of generality that $D_i \times D_j$ consists exclusively of arcs of D_c whenever $1 \leq i < j \leq k$.

LEMMA 8.4. *If $A \in \mathcal{Q}''$, $B \in \mathcal{Q}'' \cup \mathcal{Q}'$, and $A \neq B$, then $A \sim B$ iff A and B cover the same endpoint of v_0 in B.*

PROOF. Since $A \in \mathcal{Q}''$, A contains two vertices u and w that each have uniform relationships to all members of D_1, but such that u's relationship to D_1 is different from w's.

Case 1: $\{u\} \times D_1 \subseteq G_n$ *and* $\{w\} \times D_1 \subseteq G_c$. Flipping A results in $\{u\} \times D_1 \subseteq G_c$ and $w \times D_1 \subseteq G_n$. In one of these cases, A and B cover the same endpoint of v_0, so $A \sim B$ in one of these cases, by Lemma 8.3. In this case, either every member of $B \times D_1 \subseteq G_c$ or every member of $B \times D_1 \subseteq G_n$, but this contradicts membership of B in $\mathcal{Q}$. We conclude that Case 1 cannot occur.

Case 2: $\{u\} \times D_1 \subseteq G_1$ *and* $\{w\} \times D_1 \subseteq G_c$. Suppose that we flip B as necessary so that it covers the same endpoint as A does. If $A \preceq B$ now applies, then $B \times D_1 \subseteq G_c$, contradicting B's membership in $\mathcal{Q}$. By Lemma 8.3, $B \preceq A$, so no member of $B \times D_1 \in G_c$. By B's membership in $\mathcal{Q}' \cup \mathcal{Q}''$, there are pairs $(a, b), (c, d) \in B \times D_1$ such that $ab \in G_n$ and $cd \in G_1$ (where it is possible that $a = c$ or $b = d$). Flipping B so that it does not contain the same endpoint of v_0 as A results in $ab \in G_c$, and $cd \in G_1$, and $B \not\sim A$. We conclude that the claim applies in Case 2.

Case 3: $\{u\} \times D_1 \subseteq G_1$ *and* $\{w\} \times D_1 \subseteq G_n$. The proof is identical to that of Case 2, except that the roles of G_c and G_n are swapped. □

Since every circular-arc realizer has a left-right mirror image, we may assume without loss of generality that C_1'' covers the right endpoint of v_0 in R. Lemma 8.4 gives a simple criterion for flipping the remaining members of $\mathcal{Q}' \cup \mathcal{Q}''$ so that they cover the same endpoint of v_0.

8.3.2. *Case B*: $\mathcal{Q}''$ *is empty.* If $\mathcal{Q}''$ is empty, then by mirror symmetry, we can assume without loss of generality that C_1' covers the right endpoint of v_0.

Our strategy is the following (see Figure 13). We find a vertex $c_1 \in C_1'$ that distinguishes members of D_1. Suppose that c_1 covers the right endpoint of v_0 in some circular-arc realizer R. Since c_1 is the only vertex in $\{c_1\} \cup D_1$ that intersects v_0, c_1 is a source (extreme leftmost interval) in the corresponding interval realizer $R|(\{c_1\} \cup D_1)$. We use this and Δ relationships involving edges incident to c_1 to find the interval realizer $R|P_i$ for some subset P_i of arcs of D_1. In the illustration the arcs other than v_0 and c_j represent P_i. In C_j' there is a vertex c_j that distinguishes members of D_1. We are able to

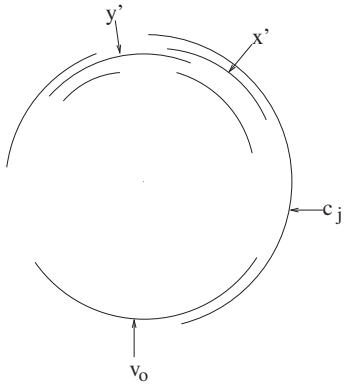


Fig. 13. The strategy for finding whether C_j' covers the same endpoint as C_1' when $\mathcal{Q}''$ is empty.

construct P_i so that c_j distinguishes members x' and y' of P_i, where $x'y' \in G_{1n}$. Now c_j is either a source or sink in the interval realizer $R|(\{c_j\} \cup D_1)$, and which of these cases applies is forced by a Δ relationship involving c_j, x', and y'. Whether it is a source or a sink tells whether c_j covers the left endpoint or the right endpoint of v_0. In the illustration c_j must cover the right endpoint of v_0, because it contains x' and overlaps y'. It could not realize those relationships if it covered the left endpoint of v_0 and entered the top part of the circle clockwise.

At certain points, we will have an interval orientation of $T|Y$ for some $Y \subseteq W$, and a vertex x that we know can be added either as a source or sink to produce an interval orientation of $T|(Y \cup \{x\})$. Which case applies determines whether x must be flipped. The interval realizer of the interval orientation of $T|Y$ is available. Rather than checking the Δ relationships between edges of G_{1n} incident to x and edges of $G_{1n}|Y$ directly, we work with an interval representation of $T|Y$, and use Algorithm 8.5:

ALGORITHM 8.5. Checking whether x can be added as a source to an interval orientation of $T|Y$, given the interval realizer R_Y of the orientation.

In the ordered-list representation of R_Y, find the first point p in the list that is to the right of all left endpoints of neighbors of x in $G(T|Y)$ and to the right of all right endpoints of neighbors of x in $(D_c)^T|Y$. If x can be added as a source, $l(x)$ must be inserted at the front of the list, and $r(x)$ must be inserted at p. The test consists of checking that no non-neighbor of x has a left endpoint to the left of p, and no overlap neighbor of x has a right endpoint to the left of p. The first check may be accomplished by counting the intervals with left endpoint to the left of p and halting if the count exceeds the degree of x in $T|(Y \cup \{x\})$. The second check is conducted similarly.

To find whether x can be added as a sink, it suffices to use the mirror transpose of R_Y to model the transpose of the orientation of $T|Y$, and test whether x can be added as a source to it. The running time of Algorithm 8.5 is $O(|N(G(T), x)|)$.

Let R be a circular-arc realizer of T_2. Let c_1 be a member of C_1' that distinguishes members of D_1, let c_j be a member of C_j' that distinguishes members of D_1, let $\mathcal{P}$ be $\{v_0\}$ and the maximal modules of $T_2|(\{v_0, c_1\} \cup D_1)$ that do not contain v_0, and let P be any set consisting of one representative from each member of $\mathcal{P}$. Let $\mathcal{P}' = \mathcal{P} - \{\{v_0\}, \{c_1\}\}$, and let $P' = P - \{v_0, c_1\}$.

We may find $R|P'$ as follows. $T_2|(\{v_0, c_1\} \cup D_1)$ is an interval matrix with interval realizer $R|(\{v_0, c_1\} \cup D_1)$. In $R|(\{v_0, c_1\} \cup D_1)$, v_0 is the interval with the leftmost left endpoint, hence a source in the corresponding interval orientation of $T_2|(\{v_0, c_1\} \cup D_1)$. We call `Partition` (Algorithm 6.22) on the initial partition $\{\{v_0\}, \{\{c_1\} \cup D_1\}\}$ in $T_2|(\{v_0, c_1\} \cup D_1)$, halting when each part is a module. The final partition is $\mathcal{P}$, and by Lemma 6.24, part 1, the ordering of its parts gives the unique interval orientation of $(T_2|(\{v_0, c_1\} \cup D_1))/\mathcal{P}$ that is consistent with $\{v_0\}$ being a source. By Lemma 6.2, the corresponding interval realizer R_1 is unique. This quotient is isomorphic to $T_2|P$, which is realized by $R|P$. In $R|P$, v_0 is also leftmost, so $R|P$ is isomorphic to R_1. By restriction, this gives $R|P'$.

If there exists $Z \in \mathcal{P}'$ whose members have differing relationships to c_j, let $x_1, x_2 \in Z$ be two vertices whose relationships to c_j differ. Let P_1 be x_1 and one arbitrary representative of each member of $\mathcal{P}' - \{Z\}$, and let P_2 be the result of replacing x_1 in P_1

with x_2. One of these will serve as P_i. If Z does not exist, let $P_1 = P_2$ consist of one representative of each member of $\mathcal{P}'$. The algorithm for finding $R|P$ works identically on $R|P_1$ and $R|P_2$, so $R|P_1$ and $R|P_2$ are isomorphic to $R|P'$, and we do not have to recompute these interval realizers for each C'_j. The relationships of c_j, $R|P_1$, and $R|P_2$ may differ, however.

LEMMA 8.6. *c_j can be added as a source to both $R|P_1$ and $R|P_2$ iff C'_1 and C'_j cover the same endpoint of v_0.*

PROOF.

Case 1: *Z does not exist.* Since c_j distinguishes members of D_1, it distinguishes $X \in \mathcal{P}'$ from $Y \in \mathcal{P}'$. Since $G_{1n}|D_1$ is connected, there exist $X', Y' \in \mathcal{P}'$ such that $X' \times Y' \subseteq G_{1n}$. Let x' and y' be the representatives of X' and Y' in P_1. Without loss of generality, suppose x' precedes y' in $R|P_1$.

If c_j covers the same endpoint as c_1 in R, then c_j has the leftmost left endpoint in the interval realizer $R|(\{c_j\} \cup P_1)$, so c_j can be added as a source to $R|P_1$, and the relationship of c_j to x' is stronger than its relationship to y'. If c_j is flipped so that it covers the opposite endpoint of v_0 from c_1, $R|(\{c_j\} \cup P_1)$ is still an interval realizer, but with c_j rightmost. Its relationship to x' becomes weaker than its relationship to y'. Since x' is earlier than y' in $R|(\{c_j\} \cup P_1)$, c_j can no longer be added as a source to $R|P_1$.

Case 2: *Z exists.* Then c_j distinguishes $x_1, x_2 \in Z$. Since D_1 is connected and $\mathcal{P}'$ consists of modules of $T_2|(\{v_0, c_1\} \cup D_1)$, there exists Z' in $\mathcal{P}'$ such that $Z \times Z' \subseteq G_{1n}$. For any y in Z', either c_j has different relationships to x_1 and y or to x_2 and y. Without loss of generality, suppose it has different relationships to x_1 and y. Now x_1 and y have the same properties as x' and y' from Case 1, so c_j can be added as a source to $R|P_1$ iff c_j and c_1 cover the same endpoint of v_0.

The results now follows from Lemma 8.2. □

Lemma 8.6 and Algorithm 8.5 provide an efficient test for deciding how to flip members of $\mathcal{Q}' - \{C'_1\}$ so that they cover the same endpoint of v_0 as C'_1 does, namely, the right endpoint of v_0.

8.3.3. *Showing that T'_3 serves as T_3.* In T'_3 all arcs in members of $\mathcal{Q}'$ and $\mathcal{Q}''$ cover the right endpoint of v_0 in some realizer R_c of T'_3. With the exception of arcs in $\mathcal{A}$, this is all arcs that have an endpoint in $A(D_1)$.

LEMMA 8.7. *There is a circular-arc realizer of T'_3 where all members of $\mathcal{Q}'$, $\mathcal{Q}''$, and $\mathcal{A}$ cover the right endpoint of v_0.*

PROOF. Let A be a member of $\mathcal{A}$ that covers the left endpoint of v_0 in T'_3. By the definition of $\mathcal{A}$, $A \times D_1 \subseteq G_1$. Therefore, $A \times D_i \subseteq G_1$ for any $i > 1$, since $D_i \times D_1 \subseteq (D_c)^{\mathrm{T}}$, and D_1 lies in the intersection of all arcs in D_i, which are, in turn, disjoint from both endpoints of v_0. Members of $\mathcal{A}$ are connected components of $\overline{G_{1n}}$. The edges of G_1 are unaffected by flipping, so flipping the members of A modifies $T'_3|A$ without affecting other relationships in T'_3.

Since they are components of $\overline{G_{1n}}$, the members of $\mathcal{A}$ are children of the root of $MD(T)$. By Lemma 6.18, for each $A \in \mathcal{A}$, the left endpoints of A are consecutive and the right endpoints of A are consecutive.

If the members of A cover the left endpoint of v_0, the only effect on the intersection matrix of flipping the members of A on the intersection matrix for the realizer is to reverse the directions of directed edges corresponding to $D_c|A$. This effect can then be undone without other side effects, by reversing the order of left endpoints of A and the order of right endpoints of A within their respective segments of the circle. The net effect is to move only those arcs that belong to A so that they cover the right endpoint of v_0, without changing the intersection matrix for the realizer.

Doing this to each $A \in \mathcal{A}$ that covers the left endpoint of v_0 produces a realizer of T_3' that satisfies the claim in the lemma. □

Lemma 8.7 justifies letting $T_3 = T_3'$. The following is now immediate:

LEMMA 8.8. *There exists a circular-arc realizer of T_3 where every member of U with an endpoint inside the region of the circle occupied by D_1 covers the right endpoint of v_0.*

8.4. *Turning T_3 into T_4.* The following corollary to Lemma 8.8 is now obvious:

COROLLARY 8.9. *In such a realizer, let p be the far endpoint of the interval $A(D_1)$ as one travels counterclockwise from the right endpoint of v_0. An arc contains p in its interior iff it is one of the following*:

- *A member of $D_2, \ldots, D_k$.*
- *A member x of U such that $x \times D_1 \subseteq D_c$.*

We may obtain T_4 from T_3 by flipping all arcs that contain p according to this corollary. This vacates the circle just to the right of p, so T_4 is an interval matrix.

8.5. *Linear-Time Implementation.* For the analysis of the time bound, we must not assume that the input graph is a circular-arc graph.

Theorems 7.11 and 7.12 ensure that we do not spend more than linear time finding neighborhood containments, even if G is not a circular-arc graph.

We use a labeling of edges of $G(T)$ with their intersection types for a sparse representation of T. We may use an array to represent all intersection types in $U \times V$. Since U has $O(m/n)$ members, the size of this array is $O(m)$.

Turning T_0 into T_1 in linear time is trivial.

To turn T_1 into T_2, we must find the bipartition of the components of $G_{2n}|N[v_0]$. This takes $O(|N[v_0]|^2)$ time using breadth-first search on the array, which is $O(m)$ since $|N[v_0]|$ is $O(m/n)$.

To turn T_2 into T_3', we must compute the $\preceq$ relation in U in linear time in Case A. For $x, y \in U$ in T_2, $x \preceq y$ iff $N[G_c, x] \cap W \subseteq N[G_c, y] \cap W$ and $N[x] \cap W \subseteq N[y] \cap W$ in T_2. Since $U \cup \{v_0\}$ is a clique in $G(T_2)$, $N[x] \cap W \subseteq N[y] \cap W$ iff $N[x] \subseteq N[y]$. Theorem 7.11 gives this in linear time.

To evaluate $N[G_c, x] \cap W \subseteq N[G_c, y] \cap W$ for each pair x, y of neighbors of v_0, note that if all members of $N(v_0)$ are flipped, $N(v_0)$ remains a clique, and the edges of G_n and G_c that are incident to each member of $N(v_0)$ are swapped. $N[G_c, x] \cap W \subseteq N[G_c, y] \cap W$ iff $N[x] \cap W \subseteq N[y] \cap W$ after $N(v_0)$ is flipped. We get the latter with Theorem 7.11 in linear time. Flipping $N(v_0)$ takes $O(n(m/n)) = O(m)$ time.

In Case B we must perform a vertex partition on a submatrix that is an interval matrix, and find a realizer for the resulting modular quotient. We have already given a linear-time algorithm for this problem, in Section 6. The submatrix is $T|(\{v_0\} \cup \{c_1\} \cup D_1)$. The size of $G(T)|(\{v_0\} \cup \{c_1\} \cup D_1)$ is linear in the size of the input graph to the recognition problem. If the algorithm does not give rise to an interval representation of the quotient, G is not a circular-arc graph, and we halt. It takes linear time to find, for each $X \in \mathcal{P}'$, another member $t(X) \in \mathcal{P}'$ that is adjacent to X in G_{1n}. Finding Z whose members are distinguished by c_j, if Z exists, takes $O(|N(c_j)|)$ time using an operation similar to a pivot on c_j. Then $t(Z)$ gives the neighbor y of x_1 and x_2 that is needed in Case 2 of the proof of Lemma 8.6.

We also have to evaluate for $c_j \in U$ whether c_j can be added as a leftmost interval to two realizers R_1 and R_2, and repeat this test after we flip c_j. This takes $O(n)$ time using Algorithm 8.5, whether or not c_j is flipped. The total time spent on this test is $O(n)$ times $O(m/n)$ vertices in U. We halt if c_j can be added as neither a leftmost nor a rightmost interval.

Flipping a member of U takes $O(n)$ time, and each member of U is flipped $O(1)$ times. The total time spent flipping neighbors of v_0 is therefore $O(m)$. Since only neighbors of U are flipped to obtain T_3 from the initial intersection matrix, $G(T_3)$ has $O(n(m/n)) = O(m)$ edges.

To turn T_3 into T_4, we must flip members of U that contain $A(D_1)$, which again takes linear time. If $\{D_2, \ldots, D_k\}$ is nonempty, we also flip every vertex in this set in T_3. If T_3 is a circular-arc matrix, then because $D_i \times D_{i+1} \subseteq D_c$ for each i from 1 to $k-1$, every arc in D_i is contained in the intersection of arcs in D_{i+1}, so D_{i+1} is a clique. Since D_c is transitive, $\bigcup\{D_2, \ldots, D_k\}$ is a clique in $G(T_2)$, $\bigcup\{D_2, \ldots, D_k\} \times D_1 \subseteq G(T_3)$. The only non-neighbors of these vertices are in $N[v_0]$, so they have degree $\Omega(n - m/n) = \Omega(n - (n-1)/2) = \Omega(n)$. We may check that this is the case in linear time, and halt if it is not. Otherwise, flipping each of them takes time proportional to its degree, hence $O(m)$ for all of them collectively.

The flip of vertices in $D_2, \ldots, D_k$ can only increase the size of $G(T_4)$ over $G(T_3)$ by adding edges between these vertices and U, and therefore adds $O(m)$ edges to $G(T_4)$. The size of $G(T_4)$ is $O(n + m)$.

9. Future Work. We have used the idea of Δ modules and their dual forcing Δ relation in [21] to obtain $O(n + m \log n)$ bounds for recognition of probe interval graphs where the partition $\{P, N\}$ of vertices into probes and nonprobes is given. This is the form of the problem that arises in genomics, and that gave rise to the initial interest in the graph class.

The previous best bound was $O(n^2)$. The strategy is to produce an interval realizer of $G|P$ that can be extended to a probe interval model of G by adding intervals for the nonprobes. Only a subset of interval realizers of $G|P$ have this property.

To do this, we use a variation of the interval matrix on G, which gives a labeling of edges of G according to the corresponding intersection types in a probe interval model. We use a variation of the Δ modules, called *extensible* Δ *modules* on $T|P$. Just as the Δ modules are a subset of the modules, the extensible Δ modules on $T|P$ are a subset of the Δ modules on $T|P$; the difference reflects the additional constraint that an interval model on $T|P$ must be able to be extended to a probe interval model of T. Their dual, the *extensible* Δ *relation*, gives all ways of producing an interval model of $T|P$ that has this property.

It remains to try to get the bound down from $O(n + m \log n)$ to linear. There may be other applications of variations on Δ modules in graphs that are defined by intersection models.

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