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On cliques of Helly Circular-arc Graphs

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Abstract

A *circular-arc graph* is the intersection graph of a set of arcs on the circle. It is a *Helly circular-arc graph* if it has a *Helly model*, where every maximal clique is the set of arcs that traverse some *clique point* on the circle. A *clique model* is a Helly model that identifies one clique point for each maximal clique. A Helly circular-arc graph is *proper* if it has a Helly model where no arc is a subset of another. In this paper, we show that the clique intersection graphs of Helly circular-arc graphs are related to the proper Helly circular-arc graphs. This yields the first polynomial (linear) time recognition algorithm for the clique graphs of Helly circular-arc graphs. Next, we develop an $O(n)$ time algorithm to obtain a clique model of Helly model, improving the previous $O(n^2)$ bound. This gives a linear-time algorithm to find a proper Helly model for the clique graph of a Helly circular-arc graph. As an application, we find a maximum weighted clique of a Helly circular-arc graph in linear time.

Keywords: algorithms, Helly circular-arc graphs, proper Helly circular-arc graphs, clique graphs, maximum weight cliques.

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1 Introduction

If $G = (V(G), E(G))$ is a graph, we denote by n and m the values of $|V(G)|$ and $|E(G)|$. A *complete set* is a subset of pairwise adjacent vertices, while a *clique* is a maximal complete set. If the vertices of G are weighted, let us say that G is a *weighted graph*. The *weight* of a clique is the sum of the weights of its vertices. The *clique graph* $K(G)$ of G is the intersection graph of its cliques.

A graph is a *clique graph* if it is isomorphic to $K(G)$ for some graph G [5,8]. Clique graphs of several classes have been characterized and several algorithms are known for testing if a graph is a clique graph of some class. Some of these can be recognized in polynomial time [11]. However, in the last year the complexity of recognition of clique graphs of arbitrary graphs was proved to be NP-Hard [1].

A *circular-arc* (CA) model $\mathcal{M}$ is a pair $(C, \mathcal{A})$, where C is a circle and $\mathcal{A}$ is a collection of arcs of C. When traversing the circle C, we will always choose the clockwise direction. If s, t are points of C, write (s, t) to mean the arc of C defined by traversing the circle from s to t. Call s, t the *extremes* of (s, t), while s is the *beginning point* and t the *ending point* of the arc. For $A \in \mathcal{A}$, write $A = (s(A), t(A))$. The *extremes* of $\mathcal{A}$ are those of all arcs in $\mathcal{A}$. Without loss of generality, we assume that all arcs of C are open arcs, no two extremes coincide, and no single arc covers C. We will say that $\epsilon > 0$ is *small enough* if ϵ is smaller than the minimum arc distance between two consecutive extremes of $\mathcal{A}$.

When no arc of $\mathcal{A}$ contains any other, $\mathcal{M}$ is a *proper* circular-arc (PCA) model. When every set of pairwise intersecting arcs share a common point, $\mathcal{M}$ is called a *Helly* circular-arc (HCA) model. If no two arcs of $\mathcal{A}$ cover C, then the model is called *normal*. A *proper Helly* circular-arc model (PHCA) is one which is both HCA and PCA. Finally, an *interval model* is a CA model where $\bigcup_{A \in \mathcal{A}} A \neq C$. A *CA (PCA) (HCA) (PHCA) (interval) graph* is the intersection graph of a CA (PCA) (HCA) (PHCA) (interval) model. A graph is *K(HCA)* (*K(PHCA)*) if it is the clique graph of some HCA (PHCA) graph. Two CA models are *equivalent* when they have the same intersection graph.

Denote by $\mathcal{A}(p)$ the collection of arcs that contain p. Clearly, the vertices corresponding to $\mathcal{A}(p)$ form a complete set. If this set is a clique, then p is called a *clique point*. For points p, p' on the circle, p *(properly) dominates* p' if $\mathcal{A}(p')$ is (properly) contained in $\mathcal{A}(p)$. When $\mathcal{A}(p) = \mathcal{A}(p')$ then p, p' are *equivalent*. Point p is a *complete point* if it is not properly dominated by any other point. In HCA graphs there is a one-to-one correspondence between cliques and non-equivalent complete points. An *intersection segment* (s, t) is a pair of consecutive extremes where s is a beginning point and t is an ending point. Points inside intersection segments are called *intersection points*. Every complete point is an intersection point, but the converse is not necessarily true [4], because there can be multiple intersection segments that are contained in exactly the same set of arcs. However, when $\mathcal{M}$ is a PHCA model, then every intersection point is also a complete point. A *complete (intersection) (clique) point representation* of $\mathcal{M}$ is a maximal set of non-equivalent complete (intersection) (clique) points. Let $I = (s(A_i), t(A_j))$ be an intersection segment and $p \in I$. The *arc reduction* of p is the arc $(s(A_i), t(A_k))$ where $A_k \in \mathcal{A}(p)$ and $t(A_k)$ is the ending point farthest from p when traversing C. Observe that when $\mathcal{M}$ is PHCA then the arc reduction of p is A_i. Let Q be a clique (complete) point representation, the *clique (complete) model* (with respect to Q) is the model formed by the arc reductions of Q. In particular, any clique model of an HCA graph G is a PCA model of $K(G)$ [4].

Circular-arc graphs and its subclasses have been receiving much attention recently ([2,10]). For CA, PCA, HCA and PHCA graphs, there are several characterizations and linear linear time recognition algorithms which construct a model (see [6,7]). In [3], K(HCA) graphs are studied. It is proved that K(HCA) graphs are both PCA and HCA graphs. In the same paper some characterizations are shown, but these characterizations did not lead to a polynomial time recognition algorithm. On the other hand, in [4] an $O(n^2)$ time algorithm that outputs the clique graph of an HCA graph is described.

In Section 2, we prove that the class of K(HCA) graphs is a "small" superset of the class of PHCA graphs. A PHCA model can be obtained in $O(n)$ time from any PCA model of a PHCA graph [7], so PHCA graphs can be recognized in linear time. With this we obtain a linear time recognition algorithm for K(HCA) graphs. In the last section, we also describe a new simple linear time algorithm for constructing a clique model of an HCA graph. In fact, we describe a more general linear time algorithm that finds a maximal dominating subset of some set of points in a CA model. This algorithm can be easily extended to find the weighted clique graph of an HCA graph, solving the maximum weighted clique problem in linear time. For CA graphs, the maximum weight clique problem can be solved in $O(n \log n + m \log \log n)$ time [9].

2 Characterization of K(HCA) graphs

Theorem 2.1 [4] *Let G be a PHCA graph. Then $K(G)$ is PHCA and every* *complete point model of G is a PHCA model of $K(G)$.*

Theorem 2.2 [7] *Let G be a PCA graph. Then G is a PHCA graph if and only if G contains neither W_4 nor 3-sun as induced subgraphs.*

Theorem 2.3 *Let G be a graph. Then $G = K(H)$ for some PHCA graph H if and only if G is a PHCA graph.*

Proof (Sketch). Consider a PHCA graph H. Graph $G = K(H)$ is PCA [3], and it is not a difficult task to check that G contains no 3-sun nor W_4 as an induced subgraph. Hence, by Theorem 2.2, G is PHCA. For the converse, let G be a PHCA graph. If G is a proper interval graph, then the result follows (see [11]). Otherwise, let $\mathcal{M} = (C, \mathcal{A})$ be a PHCA model of G and assume that it is also normal by [7]. By Theorem 2.1, it suffices to find a PHCA supermodel of $\mathcal{M}$ whose clique model is $\mathcal{M}$. Let $\mathcal{Q}$ be the set of arc reductions of $\mathcal{A}$ and $\mathcal{N} = \mathcal{A} \setminus \mathcal{Q}$. Observe that since $\mathcal{M}$ is PHCA, $\mathcal{Q}$ is a subset of $\mathcal{A}$. Also note that since G is not an interval graph then every arc of $\mathcal{A}$ contains at least one ending point of some other arc. Now, fix a small enough ϵ. For each arc $A_i \in \mathcal{N}$ let B_i be the arc $(s(A_j) - \epsilon, s(A_i) + \epsilon)$ where $t(A_j)$ is the first ending point that appears when traversing C from $s(A_i)$. If two arcs B_i, B_j share their beginning points, then modify one of them so that none of them is included in the other. We claim that $\mathcal{M}' = (C, \mathcal{A} \cup \{B_i : A_i \in \mathcal{N}\})$ is PHCA and has $\mathcal{M}$ as a clique model. To observe this, check that $\mathcal{M}'$ is a PHCA model and that the set of arc reductions of $\mathcal{M}'$ is precisely $\mathcal{A}$. □

Theorem 2.4 *Let G be a graph and U the set of universal vertices of G. Then $G = K(H)$ for some HCA graph H if and only if:*

(i) G is a PHCA graph or

(ii) $G \setminus U$ is a co-bipartite PHCA graph and $|U| \geq 2$.

To check whether a graph is PHCA can be done in linear time [7]. Thus, the recognition problems for K(HCA) and K(PHCA) graphs can be solved in linear time.

3 Construction of a clique model of an HCA graph

In [4] an $O(n^2)$ algorithm for constructing a clique model of an HCA model is described. The algorithm consists of two well defined procedures: 1) Find a clique point representation Q of the model and 2) build the clique model with respect to Q. The first procedure is the bottleneck of the algorithm, and takes $O(n^2)$ time, while the second procedure can be done in $O(n)$ time. In this section we develop a linear-time algorithm that reduces the bottleneck step to $O(n)$. Given a set P of points in an arbitrary CA model, the algorithm finds a minimum subset of the points that dominates all members of P in $O(n+|P|)$ time. Letting P be one intersection point in each of the $O(n)$ intersection segments solves the bottleneck step. From now on, let $\mathcal{M} = (C, \mathcal{A})$ be a CA model.

The *ascendant (descendant) semi-dominating sequence* of P is the subsequence $SD^+(P) = \{p_i \in P : \mathcal{A}(p_i) \not\subseteq \mathcal{A}(p_j)$ for all $p_j \in P$ and $j > i\}$ ($SD^-(P) = \{p_i \in P : \mathcal{A}(p_i) \not\subseteq \mathcal{A}(p_j)$ for all $p_j \in P$ and $j < i\}$).

Lemma 3.1 *Let $\mathcal{M} = (C, \mathcal{A})$ be a CA model and $P = \{p_1, \ldots, p_k\}$ be a sequence of circularly ordered points from C. Then both $SD^-(SD^+(P))$ and $SD^+(SD^-(P))$ are P-dominating sequences.*

Algorithms to find SD^+ and SD^- are symmetric. We describe the one to find SD^+. The algorithm works by induction on the size of a $P_i = \{p_1, p_2, \ldots, p_i\}$. After step i, we have a partition of $SD^+(P_i)$ into the following two sets:

- The members D_i that are contained in some arc that is a subset of the open interval (p_k, p_i), hence that cannot be dominated by any arc in $\{p_{i+1}, p_{i+2}, \ldots, p_k\}$. These are already known to be members of $SD^+(P)$.
- The set $Q_i = SD^+(P_i) - D_i$ in clockwise order $(q_1, q_2, \ldots q_j)$ of appearance in $[p_1, p_i]$. Their status as members of $SD^+(P)$ is uncertain; though they are in $SD^+(P_i)$, they might be dominated by points in $\{p_{i+1}, p_{i+2}, \ldots p_k\}$.

After step k, the algorithm returns $SD^+(P) = D_k \cup Q_k$. It remains to describe how to obtain D_{i+1} and Q_{i+1} from D_i and Q_i. We first find D_{i+1} by finding the members of Q_i that must be added to D_i. We need only consider the effect of arcs that begin in (p_k, p_{i+1}) and that end in $[p_i, p_{i+1})$, since arcs in (p_k, p_{i+1}) that end earlier have already been considered in determining D_i. Of these arcs, let D be the one that reaches farthest to the left, that is, whose beginning point is closest to p_k; D is the one that covers the most members of Q_i, so $D_{i+1} - D_i$ is just the points of Q_i that are covered by D.

The remaining members $Q_{i+1} = SD^+(P_{i+1}) - D_{i+1}$ are just the members of $Q_i - D_{i+1}$ that aren't dominated by p_{i+1}. A point $p \in Q_i - D_{i+1}$ is in Q_{i+1} if and only if it is contained in an arc A that doesn't contain p_{i+1}, since otherwise it would be dominated by p_{i+1}, and that contains p_k, since otherwise it would already be identified as a member of D_{i+1}. Of all such arcs, let D be the one that reaches farthest to the right of p_k; since this is the one that covers the most members of $Q_i - D_{i+1}$, Q_{i+1} is just the points of $Q_i - D_{i+1}$

that are contained in D.

It is easy to see that the algorithm can be implemented to run in $O(n+|P|)$ time using elementary techniques, since, at each step, the points of Q_i that are moved to D_{i+1} or discarded are a suffix of $(q_1, q_2, ..., q_j)$.

Theorem 3.2 *Let $\mathcal{M} = (C, \mathcal{A})$ be an HCA model of a graph G. Then a PCA (HCA) model of $K(G)$ can be found in $O(n)$ time.*

Theorem 3.3 *Let $\mathcal{M} = (C, \mathcal{A})$ be an HCA model of a graph G. Then the maximum weight clique of G can be found in $O(n)$ time.*

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