On Recognition of Threshold Tolerance Graphs and their Complements

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Abstract. A graph $G = (V, E)$ is a *threshold tolerance* graph if each vertex $v \in V$ can be assigned a weight w_v and a tolerance t_v such that two vertices $x, y \in V$ are adjacent if $w_x + w_y \geq \min(t_x, t_y)$. Currently, the most efficient recognition algorithm for threshold tolerance graphs is the algorithm of Monma, Reed, and Trotter which has an $O(n^4)$ runtime. We give an $O(n^2)$ algorithm for recognizing threshold tolerance and their complements, the threshold tolerance (co-TT) graphs, resolving an open question of Golumbic, Weingarten, and Limouzy.

1 Introduction

Tolerance graphs are an important subclass of perfect graphs that generalizes both interval graphs and permutation graphs [8]. They have been written about extensively and they model constraints in various combinatorial optimization and decision problems [8, 9, 10]. They have a rich structure and history, and interesting relationships to other graph classes. For a detailed overview of the class, see [10].

A graph $G = (V, E)$ is *threshold tolerance* if each vertex $v \in V$ can be assigned a weight w_v and a tolerance t_v such that two vertices $x, y \in V$ are adjacent when $w_x + w_y \geq \min(t_x, t_y)$ [13]. When the tolerances of the vertices are all the same, we obtain the subclass of *threshold graphs* [4].

Their complements, the *co-threshold tolerance graphs* (*co-TT graphs*), have also received attention as they have an interesting interpretation as a generalization of *interval graphs*. They are a special case of the tolerance graphs.

A graph $G = (V, E)$ is an *interval graph* if and only if each vertex $v \in V$ can be assigned an interval $I_v = [a(v), b(v)]$ on the real line such that two vertices $x, y \in V$ are adjacent exactly when their corresponding intervals intersect, in which case $\mathcal{I} = \{[a(v), b(v)] : v \in V\}$ forms an *interval model* of G. See [6, 17, 3] for surveys of the properties of this class and its relationship to other graph classes.

To illustrate the relationship of the interval graphs to the co-TT graphs, the definition can be rephrased:

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Definition 1. *A graph $G = (V, E)$ is an interval graph if and only if there exist functions $a, b : V \mapsto \mathbb{R}$ such that:*

- *$a(x) \leq b(x)$ for all $x \in V$;*
- *$xy \in E \Leftrightarrow a(x) \leq b(y) \wedge a(y) \leq b(x)$ for all $x, y \in V$.*

By this definition, $[a(x), b(x)]$ is the interval that represents x in the model. Relaxing the requirement that $a(x) \leq b(x)$, gives the class of co-TT graphs:

Definition 2. *[13] A graph $G = (V, E)$ is a co-TT graph if and only if there exist functions $a, b : V \mapsto \mathbb{R}$ such that:*

- *$xy \in E \Leftrightarrow a(x) \leq b(y) \wedge a(y) \leq b(x)$ for all $x, y \in V$.*

It is easy to see that this class is the complement of threshold tolerance graphs by setting $a(x) = w_x$ and $b(x) = t_x - w_x$ for all $x \in V$ gives functions that show that the complement is a co-TT graph, and and given the functions $a()$ and $b()$, for a co-TT graph, assigning $w_x = a(x)$ and $t_x = b(x) + w_x$ gives the weights and thresholds that show that its complement is a threshold-tolerance graph.

Following the notation in Golumbic, Weingarten and Limouzy [11], let the *blue-red partition* of V given by a co-TT model be (B, R), where $B = \{x | x \in V \text{ and } a(x) \leq b(x)\}$ and $R = \{x | x \in V \text{ and } b(x) < a(x)\}$. Given such a partition, let B be the *blue vertices* and R be the *red vertices*. The *red intervals* are the intervals $[b(x), a(x)]$ corresponding to red vertices and the *blue intervals* are the intervals $[a(x), b(x)]$ corresponding to blue vertices. Collectively, these intervals, together with their coloring, are a *co-TT model*.

The following is easily verified using Definition 2 (see Figure 1):

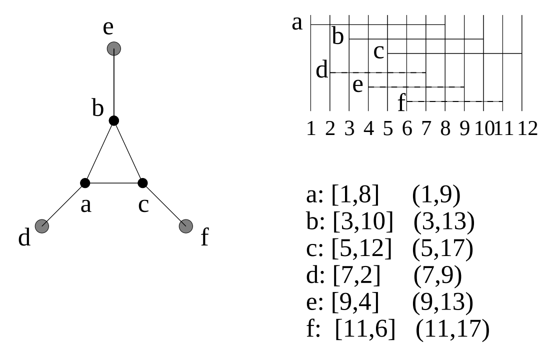


Fig. 1: Vertices that are blue in the model are black and vertices that are red in the model are gray. Though there exist models with as few as one red vertex, there are none where all vertices are blue, hence the graph is not an interval graph. The red vertices are an independent set. A red vertex is adjacent to a blue vertex if its interval is contained in the blue vertex's interval. The pairs on the lower-right illustrates the conversion of interval endpoints to weights and thresholds that represent the complement as a threshold tolerance graph.

Lemma 1. *[11] Given a co-TT model of a co-TT graph G, let (B, R) be its blue-red partition. Then:*

- *If $\{x, y\} \subseteq B$, then $xy \in E \Leftrightarrow [a(x), b(x)]$ and $[a(y), b(y)]$ intersect;*
- *If $\{x, y\} \subseteq R$, then $xy \notin E$;*
- *If $x \in B$ and $y \in R$, then $xy \in E \Leftrightarrow [b(y), a(y)]$ is contained in $[a(x), b(x)]$.*

It follows that the red vertices are an independent set. Figure 1 gives an example.

A *chord* on a cycle C in a graph is an edge not on the cycle but whose endpoints are on the cycle. A graph is *chordal* if every cycle on four or more vertices has a chord, see, for example [6]. A chord xy in an even cycle C is *odd* when the distance in C between x and y is odd. A graph is *strongly chordal* if it is chordal and every even-length of size at least six has an odd chord [5].

The following illustrates an interesting relationship between the chordal graphs, the strongly chordal graphs, the co-TT graphs and the interval graphs. A graph is chordal if and only if there is a *perfect elimination ordering*, which is an ordering $(v_1, v_2, \ldots, v_n)$ of its vertices such that for every vertex v_i, v_i and its neighbors to the right form a complete subgraph. It is easily seen that ordering the vertices of an interval or co-TT graph according to left-to-right order of $b()$ in an interval or co-TT model gives a perfect elimination ordering.

A graph is strongly chordal if and only if it has a *simple elimination ordering*, which is an ordering $(v_1, v_2, \ldots, v_n)$ such that for each v_i, the neighbors of v_i in $G[v_i, v_{i+1}, \ldots, v_n]$ are ordered by closed neighborhood containment. That is, if v_j, v_k are two such neighbors, one of $N[v_j]$ and $N[v_k]$ is a subset of the other in $G[v_i, v_{i+1}, \ldots, v_n]$. A simple elimination is a special case of a perfect elimination ordering. Another characterization is that a graph is strongly chordal if and only if it has no *strong elimination ordering*, $(v_1, v_2, \ldots, v_n)$ such that whenever v_i, v_j, v_k are three vertices and $i < j < k$, then $N[v_j] \subseteq N[v_k]$ in $G[v_i, v_{i+1}, \ldots, v_n]$. It is easily seen that ordering the vertices of an interval or co-TT graph according to left-to-right order of $b()$ in an interval or co-TT model also gives a strong elimination ordering. The following is an immediate consequence:

Theorem 1. *[13] Every co-TT graph is strongly chordal.*

A graph is co-TT if and only if it has a *proper elimination ordering*, which is an ordering $(v_1, v_2, \ldots, v_n)$ of vertices such that whenever $v_i v_j \notin E$, either v_i is to the left of all members of $N(v_j)$ or v_j is to the left of all members of $N(v_i)$. A proper elimination ordering is a simple elimination ordering and a strong elimination ordering, hence a perfect elimination ordering. It is easily seen that ordering all vertices in left-to-right order of $b()$ in an interval or co-TT model also gives this. It is shown in [13] that a graph is a co-TT graph if and only if it admits a proper elimination ordering.

To complete this taxonomy, a graph is an interval graph if and only if it admits a proper elimination ordering $(v_1, v_2, \ldots, v_n)$, whenever $v_i v_j \notin E$ and $i < j$, then v_i is to the left of all members of $N[v_j]$. Equivalently, for comparison to proper elimination orderings, this is an ordering where v_i is to the left of all members of $N[v_j]$ or v_j is to the left of all members of $N[v_i]$. Ordering the vertices in left-to-right order of $b()$ in an interval model, that is, by right endpoint, gives such an ordering. Conversely, it is easy to obtain an interval model, given such an ordering, by making j be the right endpoint of each vertex v_j and extending the left endpoint far enough to the left to meet the right endpoint of the earliest neighbor of v_j. Such an ordering is a proper elimination ordering, hence a simple elimination ordering and a perfect elimination ordering.

Note that R is an independent set, and that the blue intervals are an interval model of $G[B]$, hence $G[B]$ is an interval graph. A vertex v of a graph is *simplicial* if its neighbors induce a complete subgraph. For each $r \in R$, the intervals corresponding to neighbors of r contain r's interval, so they have a common intersection point. Since the neighbors of r are blue, r is simplicial.

Henceforth, we will denote a co-TT model as $\mathcal{I}(B, R)$, which is a set of intervals on the line, together with an implied bijection from vertices to the intervals, and where (B, R) is the blue-red partition. Suppose x maps to an interval $[l, r]$. If $x \in B$, $a(x)$ is implicitly l and $b(x)$ is implicitly r, whereas if $x \in R$, $a(x)$ is implicitly r and $b(x)$ is implicitly l.

Despite the similarities co-TT models and interval models, the best time bound for recognition of threshold tolerance and co-TT graphs until now has been $O(n^4)$ [13], whereas linear-time recognition of interval graphs has been known for some time [2].

A graph is a *split graph* if its vertices can be partitioned into a complete subgraph and an independent set. Golumbic, Limouzy and Weingartner [11] showed that *split co-TT* graphs, that is, those graphs that are both split graphs and co-TT graphs, can be recognized in $O(n^2)$ time and a forbidden subgraph characterization for split co-TT graphs was given. We generalize this bound to recognition of arbitrary co-TT graphs. The structural insight of Section 3, developed in [11] is essential to our approach. This gives an $O(n^2)$ bound for recognition of threshold tolerance graphs also, since it now takes $O(n^2)$ time to recognize whether the complement of a graph is a co-TT graph.

2 Preliminaries

Two sets *overlap* if they intersect and neither is a subset of the other.

Given a set A of directed edges, let A^T denote the *transpose* $\{(y,x)|(x,y) \in A\}$. We view undirected graph as a special case of a directed graph, where each edge xy consists of two arcs (x,y) and (y,x).

Given a binary (0,1) matrix, we treat the rows and columns as bit-vector representations of sets. A row is the set of columns where the row has a 1, and similarly for columns. This allows us to apply set operations to rows or to columns, such as evaluating whether one row is a subset of another.

An undirected graph $G = (V,E)$ is a special case of a symmetric directed graph, so we may refer to the directed edges of an undirected graph. For $\emptyset \subset V' \subseteq V$, let $G[V']$ denote the subgraph of G induced by V'. For $v \in V$, let the *open neighborhood* of v, denoted $N_G(v)$, be the set of neighbors of v in G, and let its *closed neighborhood*, denoted $N_G[v]$, be $N_G(v) \cup \{v\}$. When G is understood, we may denote these $N(v)$ and $N[v]$.

A *maximal clique* of a graph is a complete subgraph that is properly contained in no other complete subgraph. Two vertices u and v are *false twins* if $N(u) = N(v)$. Note that this implies that they are nonadjacent. They are *true twins* if $N[u] = N[v]$, which implies that they are adjacent. The pairs of false twins in a graph are an equivalence relation, as are the pairs of true twins.

Given a collection of lists of integers from $\{1, 2, \ldots, n\}$, whose sum of lengths is k, we may sort each list by numbering the lists, sorting all the elements of all the lists in a single radix sort using list number as primary sort key and element value as secondary sort key. This takes $O(n+k)$ time. We can sort the adjacency lists of a graph in $O(n+m)$ time, for example. Also, we can sort the collection of the lists lexicographically in $O(n+k)$ time even though they have different lengths [1]. The following is a consequence that we will refer to:

Proposition 1. *It takes $O(n+m)$ time to identify all equivalence classes of true twins, by sorting the closed neighborhoods lexicographically, and to find all equivalence classes of false twins, by sorting the open neighborhoods lexicographically.*

3 Reduction to the case where G is a co-TT graph and inferring a blue-red partition

We give an $O(n^2)$ algorithm that has the precondition that its input graph G is a co-TT graph and the postcondition that it has returned a valid co-TT model. The reason that this suffices for recognition is that such an algorithm must fail to return a valid co-TT model if and only if its input graph is not a co-TT graph, since no valid co-TT model exists if it is not. (Our algorithm sometimes returns an invalid model, and sometimes halts when it recognizes that G lacks a property that co-TT graphs have.) Given a graph $G = (V,E)$ and co-TT model $\mathcal{I}(B,R)$ on V, it trivially takes $O(n^2)$ time to determine whether $\mathcal{I}(B,R)$ is a valid co-TT model of G, by applying Lemma 1 to each pair of intervals and comparing the result with the corresponding adjacency-matrix entry for G. We show how to implement the algorithm so that it halts in $O(n^2)$ time, whether or not G meets its precondition.

In the rest of this paper, we assume that the precondition to the the algorithm is met, that is, that G is a co-TT graph, except when we analyze the running time in the case where G is not a co-TT graph.

A key element of our approach is the following insight, which is given by Golumbic, Limouzy and Weingartner in [11].

Lemma 2. *If G is a co-TT graph, then there exists a co-TT model where the red vertices are the simplicial vertices that have no true twins in G, and the blue vertices are all others.*

In the graph on the left of Figure 2, for example, the simplicial vertices are $\{d, d', e, e', f\}$. Those that have no true twins are e, e' and f, by the lemma, these are red in some co-TT model, and d and d' are true twins, so these are blue in some co-TT model. The model at the bottom left illustrates this. The idea behind the proof is that if a vertex is red, then it is contained in those of its blue neighbors and it has no red neighbors. The intervals of its blue neighbors must therefore have a common intersection, and they must induce a complete subgraph. A simplicial vertex that has no true twins can be inserted as a red interval in this common intersection. If it has a true twin, then at most one of the vertex and its twin can be red, since they are adjacent. However, if one of them is blue, then the other can be modeled with a second blue interval that matches that of the first.

By this, the authors implied an obvious linear-time algorithm for finding a blue-red partition in an arbitrary co-TT graph. However, since they only addressed split co-TT graphs, they did not give it explicitly. For completeness, we give it here:

Lemma 3. *If G is a co-TT graph, it takes $O(n+m)$ time to find a blue-red partition (B, R).*

Proof. It takes $O(n + m)$ time to recognize whether a graph is chordal, the sum of cardinalities of the maximal cliques of a chordal graph is $O(n + m)$ and they take $O(n + m)$ time to find [15]. Since a co-TT graph is chordal, find its maximal cliques, in $O(n + m)$ time. A vertex is simplicial if and only if it is a member of exactly one maximal clique. It takes time proportional to the sum of cardinalities of the maximal cliques to test this on all vertices, by traversing the vertices in each maximal clique, marking them, and marking each vertex as non-simplicial if it has already been marked during traversal of another maximal clique. By Proposition 1, it takes $O(n + m)$ time to identify all equivalence classes of true twins.

This gives a reduction to finding whether a graph G' has a co-TT model with a given partition (B', R'), where B' has no true twins and the vertices in R' have no false twins. The idea behind the reduction is given in Figure 2, and the reduction is given in Algorithm 1.

Lemma 4. *The reduction of Algorithm 1 is correct, and can be implemented to run in $O(n + m)$ time whether or not the input graph is a co-TT graph.*

Proof. If $\mathcal{I}(B, R)$ is a co-TT model of G, then $\mathcal{I}(B, R) \cap (B' \cup R') = \mathcal{I}(B', R')$ is a co-TT model of G'. Therefore, G' has a co-TT model with blue-red partition (B', R'). Given $\mathcal{I}'(B', R')$, correctness of the construction of a co-TT model of G is then immediate from Lemmas 1 and 3. The time bound given in the proof of Lemma 3 depends only on the input graph G being chordal, which takes $O(n+m)$ time to determine [15]. Since all co-TT graphs are chordal, G can be rejected as a co-TT graph if it is not chordal.

The motivation for the reduction is given by the following lemma, which we use to simplify the analysis:

Lemma 5. *Let G', B', R' be as in Algorithm 1. For any pair $\{x, y\}$ of distinct vertices, $N[x] \neq N[y]$, and for any pair $\{r, r'\} \subseteq R'$ $N(r) \neq N(r')$.*

Proof. The red vertices of G remain simplicial in G'. If b and b' are two members of B', then since each true-twin equivalence class of B' has only one member, $N[b] \neq N[b']$. If r is a red vertex, it has no true twins in G, hence it has no true twins in G', and $N[r] \neq N[y]$ for any other vertex y. Since r is the only red member of its false-twin equivalence class in G', $N(r) \neq N(r')$ for any other red vertex r'.

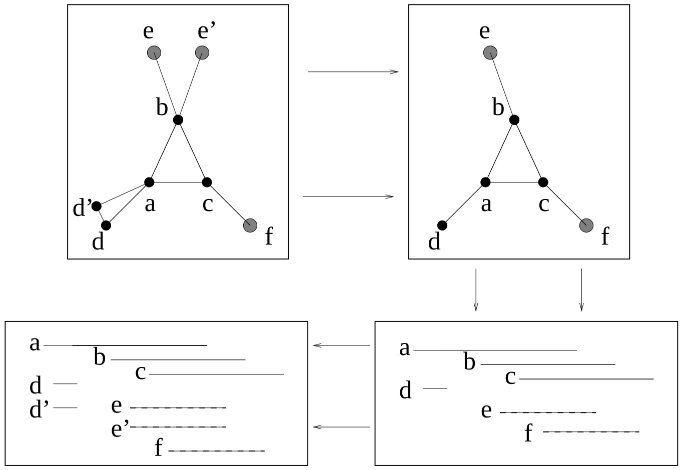


Fig. 2: The reduction of Algorithm 1. A blue-red partition is given by Lemma 2. All but one member of each equivalence class of blue twins is eliminated. In the figure, d' is eliminated from the equivalence class $\{d, d'\}$. All but one member of each equivalence class of red twins is eliminated. In the figure, e' is eliminated from the equivalence class $\{e, e'\}$. If a model can be found for the reduced graph on the right, then it can be turned into a model of the original graph by duplicating intervals from the interval corresponding to the retained member of each twin equivalence class of size greater than 1.

Data: A co-TT graph G
Result: A co-TT model of G

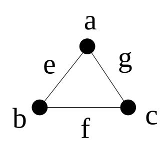
Find the blue-red partition (B, R) for some co-TT model of G (Lemma 3);
$B' \longleftarrow$ one representative from each equivalence class of true twins of B (Proposition 1);
Remove any isolated vertices from R;
$R' \longleftarrow$ one representative from each equivalence class of false twins of R (Proposition 1);
$G' \longleftarrow G[B' \cup R']$;
Find a co-TT model $\mathcal{I}'(B', R')$ of G' (Algorithm 2);
for $b \in B \setminus B'$ **do**
    Insert a blue interval for b to $\mathcal{I}'$ equal to that of the representative of b's true-twin class;
end
for $r \in R \setminus R'$ **do**
    if *r is an isolated vertex* **then**
        Insert a red interval for r that contains all blue intervals;
    end
    else
        Insert a red interval for r to $\mathcal{I}'$ equal to that of the representative of r's false-twin class;
    end
end
Return the resulting model $\mathcal{I}(B, R)$;

Algorithm 1: `Co-TT-Model`(G)

– Henceforth in the paper, we will let G', B' and R' denote these elements of the reduction of Algorithm 2, and let $V' = B' \cup R'$ denote the vertices of G'.

4 Strongly Chordal Graphs and Chordal Bipartite Graphs

An *edge-vertex incidence matrix* for a graph has one row for each vertex, one column for each edge, and a 1 in row i, column j if edge j is incident to vertex i. A binary matrix is *totally balanced* if and only if it does not have as a submatrix the edge-vertex incidence matrix of a cycle of length at least three. (See Figure 3.) The *augmented adjacency matrix* of a graph on vertex set $\{v_1, v_2, \ldots, v_n\}$ is the binary matrix that has a 1 in row i, column j if $v_j \in N[v_i]$. That is, it is the result of adding 1's on the diagonal to the adjacency matrix. The *bipartite adjacency matrix* for a bipartite graph $G = (\{v_1, v_2, \ldots, v_j\}, \{w_1, w_2, \ldots, w_k\}, E)$ is the binary matrix that has a 1 in row i, column j if w_j is a neighbor of v_i.



	e	f	g
a	1		1
b	1	1	
c		1	1

Fig. 3: The edge-vertex incidence matrix of a cycle of length three. A one in a row and column indicates that the edge of the column is incident to the vertex of the row. Omitted entries in the figure are implicitly zeros. Rows a and b and columns e and g induce a *Gamma*, which is a 2×2 binary matrix that has a 0 only in its lower right corner. No permutation of rows and columns of the matrix eliminates the existence of a Gamma. The minimal matrices such that no permutation of rows and columns eliminates the existence of Gammas are the edge-vertex incidence matrices of cycles of length greater than or equal to three. It follows that if a binary matrix is totally balanced if and only if there exists a permutation of rows and columns that is Gamma-free.

Theorem 2. *[5] A graph is strongly chordal if and only if its augmented adjacency matrix is totally balanced.*

A bipartite graph is *chordal bipartite* if every cycle of length greater than or equal to six has a chord. (See [17] for a survey.)

Theorem 3. *[7] A bipartite graph is chordal bipartite if and only if its bipartite adjacency matrix is totally balanced.*

A *doubly-lexical ordering* of a matrix is an ordering of its rows and columns such that for two rows i and j, if k is the rightmost column where the rows differ, then the row that has a 1 in column k is below the other, and for two columns i' and j', if k' is the lowest row where the columns differ, then the column with the 1 in row k' is to the right of the other. An $O(m \log n)$ algorithm for finding a doubly-lexical ordering of any binary matrix, given with a sparse representation, is given by Paige and Tarjan [14], where n is the number of rows and columns and m is the number of 1's. An $O(pq)$ variant is given by Spinrad for $p \times q$ matrices [16]. In this paper, we make use of the latter result.

Lemma 6. *Let $\{V_1, V_2\}$ be a partition of vertices of a strongly chordal graph, G. Then the bipartite graph $H = (V_1, V_2, \{xy | x \in V_1$ and $y \in V_2\})$ is chordal bipartite.*

Proof. Any cycle C in H is a cycle of G. If $|C| \geq 6$, then it is a cycle that has an odd chord in G. Since the vertices on C alternate between V_1 and V_2 around C, the odd chord has one end in V_1 and the other in V_2, hence it is a chord of C in H.

Theorem 4. *[16] Given a $p \times q$ binary matrix, it takes $O(pq)$ time to determine whether it is totally balanced, and, if so, to determine for each ordered pair (i, j) of rows whether row i is a subset of row j.*

The idea behind the proof is to use the algorithm of [16] to find a doubly-lexical ordering. From this, it is easy to identify identical rows, since they are consecutive. There is an easy linear-time test either to detect a Gamma or to determine that the matrix is Gamma-free. If it is Gamma-free, then for rows i and j with i earlier than j, if the rows are not equal, then row j cannot be a subset of row i, since j has a 1 and i has a 0 in the rightmost column where the two rows differ. To determine whether row i is a subset of row j, let h be the first column where row i has a 1. If row j has a 0 in column h, then row i is not a subset of row j. Otherwise, row i must be a subset of row j, which is seen as follows. Suppose to the contrary that it is not. Then there must be some column h' where row i has a 1 and j has a 0. Then rows i, j and columns h, h' induce a Gamma, a contradiction.

The main consequence of these results for this paper is summarized as follows:

Lemma 7. *In G', it takes $O(n^2)$ time to find whether $N[b_1] \subset N[b_2]$ for each ordered pair (b_1, b_2) of distinct vertices and whether $N(r_1) \subset N(r_2)$ for each ordered pair of distinct vertices in R'.*

Proof. The bound for closed neighborhood containments follows from Theorems 2, 1 ,4 and Lemma 5. Since R' is an independent set of G', the bound for open neighborhood containments follows from Theorems 3, 1, 4, Lemma 5, and an application of Lemma 6 to the bipartite graph $H = (R', B', E \cap (R' \times B'))$.

5 Finding a co-TT model $\mathcal{I}(B', R')$ of G'

In this section, we given an algorithm to find a co-TT model $\mathcal{I}(B', R')$ of G', where (G', B', R') are as defined in the reduction of Algorithm 2.

Definition 3. *A set $\mathcal{I}$ of n intervals on the line is in* standard form *if all endpoints are distinct and elements of $\{1, 2, \ldots, 2n\}$.*

For example, the co-TT model of Figure 1 is in standard form.

Definition 4. *Let $G = (V, E)$ be a graph. Let A_V denote $\{(x, y) | x, y \in V$ and $x \neq y\}$. Let $\mathcal{I}$ be a set of intervals in standard form. For each vertex x, assign a member $I_x \in \mathcal{I}$ to x, so that the assignment is a bijection from V to $\mathcal{I}$.*

For $(x, y) \in A_V$, let its intersection type *be an* overlap *if I_x and I_y overlap, a* non-intersection *if I_x and I_y are disjoint, a* containment *if I_x contains I_y, and a* subset relationship *if I_x is a subset of I_y. These labels are* intersection labels *of elements of A_V. Let us say that $\mathcal{I}$* realizes *the assignment of these labels. Let $E_n(\mathcal{I})$ denote the set of elements of A_V that are labeled as non-intersections, $E_o(\mathcal{I})$ those that are labeled as overlaps, $A_s(\mathcal{I})$ those that are labeled as subset relationships, and $A_c(\mathcal{I})$ be those that are labeled as containments. If $\mathcal{I}$ is understood, we may denote them E_n, E_o, A_s and A_c, respectively (see Figure 4 for an illustration).*

Lemma 8. *[12] Given a set V and an arbitrary assignment of intersection labels to the elements of A_V, it takes $O(n^2)$ time to find a set of intervals in standard form that realizes the labeling, or else to determine that no such set of intervals exists.*

The bound given in [12] is actually $O(n + k)$ where k is the number of elements of A_V that are not in E_n. When the labeling is for an interval model of a graph G, construction of the interval model takes $O(n + m)$ time, since the edges of G are those elements of A_V that are not in E_n. Unfortunately, this bound does not apply to co-TT models, which assign a variety of labels to elements of A_V that are not edges of G.

A_V	a	b	c	d	e
a	*	A_c	E_o	E_o	E_n
b	A_s	*	E_o	E_n	E_n
c	E_o	E_o	*	A_c	E_o
d	E_o	E_n	A_s	*	E_o
e	E_n	E_n	E_o	E_o	*

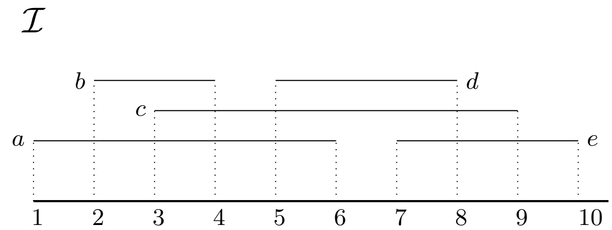


Fig. 4: A matrix representation of A_V and an interval model $\mathcal{I}$ that realizes the intersection types specified in A_V. An asterisk denotes the absence of an intersection type.

Lemma 8 further reduces the problem of finding a co-TT model $\mathcal{I}(B', R')$ to that of assigning intersection labels to elements of A_V that are consistent with some co-TT model $\mathcal{I}'(B', R')$ of G'. In the rest of this section, we show how to assign such a labeling in $O(n^2)$ time.

Definition 5. *Let $\mathcal{I}$ be an interval model of $G'[B']$ in standard form. The* red extension *of $\mathcal{I}$ is constructed as follows. (See Figure 5.) For $r \in R'$, let l_r be the rightmost left endpoint of a neighbor of r and let r_r be the leftmost right endpoint of a neighbor. Create an interval for r whose left endpoint is at $l_r + 1/3$ and right endpoint is at $r_r - 1/3$. Order each block of coinciding right red endpoints in descending order of neighborhood size and each block of coinciding left red endpoints in ascending order of neighborhood size. Put the model in standard form by listing the endpoints in left-to-right order.*

Lemma 9. *Let $\mathcal{I}$ be as in Definition 5. If the preconditions of Algorithm 2 are met, a red extension of $\mathcal{I}$ exists.*

Proof. Suppose the preconditions are met, but that a red extension fails to exist. Then $r_r < l_r$ for some $r \in R'$. Since $\mathcal{I}$ is an interval model of $G'[B']$, this implies that r has two blue neighbors that are nonadjacent to each other, and r fails to be simplicial, a contradiction.

Proposition 2. *Suppose $\mathcal{I}(B', R')$ is a co-TT model of G'. Then the red extension $\mathcal{I}'(B', R')$ of $\mathcal{I}(B', R')[B']$ is a co-TT model of G'.*

By Proposition 2, any normal co-TT model can be turned into a co-TT model by replacing the model with the red extension of its blue intervals:

Definition 6. *If G' is a co-TT graph, then a co-TT model $\mathcal{I}(B', R')$ of G' is* normalized *if $\mathcal{I}(B', R')$ is the red extension of $\mathcal{I}(B', R')[B']$.*

The algorithm for assigning the intersection labels to A_V is given as Algorithm 2, and gives a summary of the roles of the lemmas that follow in this section.

Lemma 10. *There exists a co-TT model $\mathcal{I}(B', R')$ of G' such that for each $(b_1, b_2) \in A_{B'}$:*

- *$(b_1, b_2) \in E_n$ if b_1 and b_2 are nonadjacent;*
- *$(b_1, b_2) \in A_s$ and $(b_2, b_1) \in A_c$ if $N[b_1] \subset N[b_2]$;*
- *$(b_1, b_2) \in E_o$ if b_1 and b_2 are adjacent but neither $N[b_1] \subset N[b_2]$ nor $N[b_2] \subset N[b_1]$.*

Proof. Let $\mathcal{I}'(B', R')$ be a co-TT model, and let I_1 and I_2 be the intervals for b_1 and b_2, respectively, in $\mathcal{I}'(B', R')$.

If b_1 and b_2 are nonadjacent, then since they are elements of B', I_1 and I_2 do not intersect in $\mathcal{I}'(B', R')$, by Lemma 1. This establishes the first claim.

For the second claim, suppose $N[b_1] \subset N[b_2]$. If $N[b_1] \subset N[b_2]$, then if the right endpoint of I_1 lies to the right of the right endpoint of I_2 in $\mathcal{I}(B', R')$, in violation of the claim, the

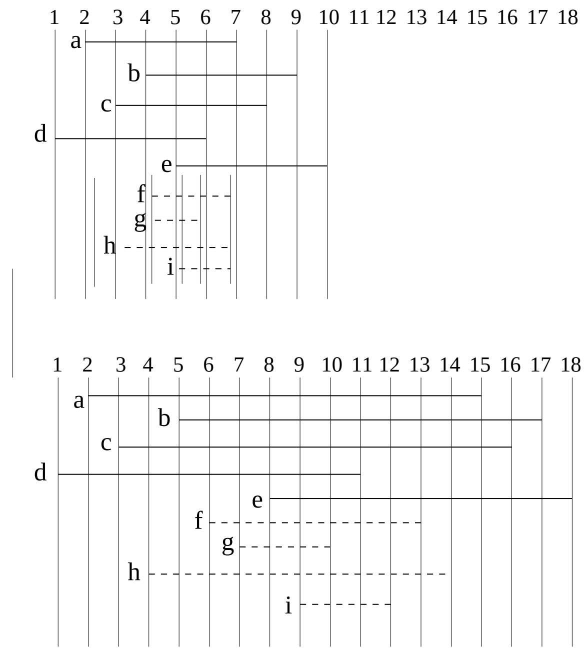


Fig. 5: The red extension of a blue model. Given a blue model (solid lines) and the adjacencies of red vertices to blue vertices, we stretch each red intervals as far as we can without losing any of its blue neighbors. The vertex f has $\{a, b, c\}$ as neighbors, so its red interval is made to fit just inside the common intersection of intervals a, b, and c. The same approach is used for other red vertices. When left endpoints of red intervals coincide, we order them in descending order of right endpoint, as in the example of $\{f, g\}$ and when right endpoints coincide, we order them in descending order of left endpoint, as in the case of $\{f, h, i\}$. Sorting all endpoints according to this ordering gives a new model in standard form. The result might not be a co-TT model: if the neighbors of f were $\{a, b\}$, it would have the same red interval in the red extension, falsely representing f as a neighbor of c. This happens when some other blue model of $G'[B']$ is has a co-TT model as its red extension.

Data: The graph $G' = (V', E')$ is a co-TT graph with blue-red partition (B', R'), B' has no true twins, and R' has no isolated vertices or false twins

Result: A labeling of elements of A_V with their intersection types in a co-TT model $\mathcal{I}(B', R')$ of G'

for $(x, y) \in A_{V'}$ **do**
    Find whether $N[x] \subset N[y]$ (Lemma 7);
end
for $(r_1, r_2) \in A_{R'}$ **do**
    Find whether $N(x) \subset N(y)$ (Lemma 7);
end
for $(b_1, b_2) \in A_{B'}$ *(Lemma 10)* **do**
    if $b_1 b_2 \notin E'$ **then**
        Assign (b_1, b_2) to E_n ;
    end
    else if $N[b_1] \subset N[b_2]$ **then**
        Assign (b_1, b_2) to A_s and (b_2, b_1) to A_c ;
    end
end
for $(b_1, b_2) \in A_{B'}$ **do**
    if (b_1, b_2) *has not already been assigned* **then**
        Assign (b_1, b_2) to E_o ;
    end
end
Construct an interval model $\mathcal{I}_B$ of $G[B']$ realizing these labels (Lemma 8) ;
Let $\mathcal{I}'$ be the red extension of $\mathcal{I}_B$;
for $(r_1, r_2) \in A_{R'}$ **do**
    if $N(r_1) \subset N(r_2)$ **then**
        Assign (r_1, r_2) to A_c and (r_2, r_1) to A_s (Lemma 11);
    end
    else if $(r_1, r_2) \in E_n(\mathcal{I}')$ **then**
        Assign (r_1, r_2) to E_n (Lemma 12);
    end
end
for $(b, r) \in B' \times R'$ **do**
    if $br \in E'$ **then**
        Assign (b, r) to A_c and (r, b) to A_s (Lemma 1);
    end
    else if $(b, r) \in A_s(\mathcal{I}')$ **then**
        Assign (b, r) to A_s and (r, b) to A_c (Lemma 12);
    end
    else if $(b, r) \in E_n(\mathcal{I}')$ **then**
        Assign $(b, r), (r, b) \in E_n$ (Lemma 12);
    end
end
for $(x, y) \in A_{V'}$ **do**
    if (x, y) *has not been assigned* **then**
        Assign (x, y) to E_o ;
    end
end
Apply Lemma 8 to find a co-TT model $\mathcal{I}(B', R')$ of G';

Algorithm 2: `Co-TT-Model`(G', B', R')

right endpoint of I_1 can be moved to be just to the left of the right endpoint of I_2 without causing b_1 to lose any neighbors of b_2 in the represented graph. Since $N[b_1] \subset N[b_2]$, this does not cause b_1 to lose any neighbors in the represented graph. By symmetry, a left endpoint of b_2 that lies outside of b_1's interval can be brought inside it without affecting the represented graph. Now (b_1, b_2) conforms to the second claim. Let us call this operation a *contraction*.

Number the blue vertices $(b_1, b_2, \ldots, b_k)$ in descending order of neighborhood size. For i from 1 to k, perform a contraction on each vertex b' such that $N[b'] \subset N[b_i]$ but whose interval is not contained in that of b_i. This brings (b', b_i) into conformity with the second claim. By induction on i, the pairs that are in violation of the second claim are confined to $\{b_{i+1}, b_{i+2}, \ldots, b_k\}$ after iteration i. There can be no violation of the second claim after iteration k.

For the third claim, since b_1 and b_2 are adjacent, $N[b_1] \cap N[b_2]$ is nonempty. Since there are no true blue twins in B', $N[b_1] \neq N[b_2]$. Therefore, $N[b_1] \not\subset N[b_2]$ and $N[b_2] \not\subset N[b_1]$ implies that $N[b_1]$ and $N[b_2]$ overlap. Since b_1 and b_2 are adjacent and blue, I_1 intersects I_2 by Lemma 1. If $x \in N[b_1] \cap N[b_2]$, then if x is red, I_1 contains x's interval but I_2 does not. I_1 is not contained in I_2. Similarly, I_2 is not contained in I_1. I_1 and I_2 intersect, but neither is contained in the other, so $(b_1, b_2) \in E_o(\mathcal{I}'(B', R'))$.

By Lemmas 8 and 10, we may now find an an assignment $\mathcal{I}_B$ of intervals to B' such that the intersection types are the same as those of $\mathcal{I}(B', R')[B']$ for some co-TT model $\mathcal{I}(B', R')$ of G' in $O(n^2)$ time. Note that $\mathcal{I}_B$ and $\mathcal{I}(B', R')[B']$ are both interval models of $G'[B']$. Since there may be many interval models satisfying these intersection types, it is not necessarily the case that $\mathcal{I}_B = \mathcal{I}'(B', R')[B']$ for any co-TT model $\mathcal{I}'(B', R')$ with blue-red partition.

– Henceforth, as in Algorithm 2, we will let $\mathcal{I}_B$ denote this interval model of $G'[B']$.

Lemma 11. *If $\mathcal{I}(B', R')$ is a normalized co-TT model of G', then for distinct members $r_1, r_2 \in R'$, $(r_1, r_2) \in A_s(\mathcal{I}(B', R'))$ if and only if $N(r_2) \subset N(r_1)$.*

Proof. Let I_1 be the interval for r_1 and I_2 be the interval for r_2 in $\mathcal{I}(B', R')$.

Since R' has no isolated vertices, neither $N(r_1)$ nor $N(r_2)$ is empty. If $(r_1, r_2) \in A_s(\mathcal{I}(B', R'))$, that is, that $I_1 \subset I_2$, then any blue interval that contains I_2 contains I_1. By Lemma 1, $N(r_2) \subseteq N(r_1)$. Since R' has no false twins, $N(r_2) \subset N(r_1)$.

Suppose that $N(r_1) \subset N(r_2)$, but r_2's interval does not contain r_1's in $\mathcal{I}(B', R')$, contradicting the claim. Then either I_1's right endpoint lies to the right of I_2's, or I_1's left endpoint lies to the left of I_2's. Assume without loss of generality that I_1's right endpoint lies to the right of I_2's. Since $\mathcal{I}(B', R')$ is the red extension of $\mathcal{I}(B', R')[B']$, I_1's right endpoint lies in the consecutive block of red endpoints to the left of the leftmost right endpoint of neighbors of r_1. Since $N(r_1) \subset N(r_2)$, the right endpoint of I_2 cannot lie to the right of this endpoint by Lemma 1. Since it lies to the right of the right endpoint of r_1, it must lie in the same block of red right endpoints as the right endpoint of r_1. By the way right endpoints in such a block are ordered in a right extension, this implies that $|N(r_2)| < |N(r_1)|$, a contradiction.

Though $\mathcal{I}_B$ realizes the intersection types of $\mathcal{I}(B', R')[B']$ for some co-TT model $\mathcal{I}(B', R')$, it may not be the case that $\mathcal{I}_B = \mathcal{I}'(B', R')[B']$ for any co-TT model $\mathcal{I}'(B', R')$ of G'. There may be many interval models of $G'[B']$ that realize these intersection types. Therefore, the red extension of $\mathcal{I}_B$ might not be a co-TT model. We can nevertheless use it to derive some of the intersection types in $\mathcal{I}(B', R')$:

Lemma 12. *Let $\mathcal{I}(B', R')$ be a normalized co-TT model of G' whose intersection types among the blue vertices are the same as those given by $\mathcal{I}_B$. Let $\mathcal{I}'$ be the red extension of $\mathcal{I}_B$.*

1. *For $r_1, r_2 \in R'$, $(r_1, r_2) \in E_n(\mathcal{I}')$ if and only if $(r_1, r_2) \in E_n(\mathcal{I}(B', R'))$;*

2. For $r \in R', b \in B'$, $(b,r) \in E_n(\mathcal{I}')$ if and only if $(b,r) \in E_n(I(B',R'))$, and $(b,r) \in A_s(\mathcal{I}')$ if and only if $(b,r) \in A_s(\mathcal{I}(B',R'))$.

Proof. For the first claim, suppose first that $N(r_1) \cup N(r_2)$ is not a complete subgraph of G'. Since r_1 and r_2 are each simplicial, r_1 has a neighbor x and r_2 has a neighbor y such that x and y are nonadjacent. Since $x, y \in B'$ and $\mathcal{I}'[B'] = \mathcal{I}_B$ and $I(B',R')[B']$ are interval models of $G'[B']$, x and y have disjoint intervals in both models. By the construction of the red extension, r_1's interval is contained in x's interval in both models and r_2's interval is contained in y's interval in both models. Therefore, r_1's and r_2's intervals are disjoint in both models, so $(r_1, r_2) \in E_n(\mathcal{I}')$ and $(r_1, r_2) \in E_n(\mathcal{I}(B', R'))$. Now, suppose to the contrary that $N(r_1) \cup N(r_2)$ is a complete subgraph. Since $\mathcal{I}'[B']$ and $\mathcal{I}(B',R')[B']$ are both interval models of $G'[B']$, and the intervals corresponding to $N(r_1) \cup N(r_2)$ have a common intersection in both models. By the definition of the red extension, r_1 and r_2 intersect in this common intersection in $\mathcal{I}'$, which is the red extension of $\mathcal{I}'_B$, and in $\mathcal{I}(B',R')$, which is the red extension of $\mathcal{I}(B',R')[B']$. Therefore, $(r_1, r_2) \notin E_n(\mathcal{I}')$ and $(r_1, r_2) \notin E_n(\mathcal{I}(B', R'))$.

For the first part of the second claim, suppose first that $N(r) \not\subseteq N(b)$. It follows that b's interval cannot intersect r's interval in $\mathcal{I}'$ or in $\mathcal{I}(B',R')$, since all neighbors of r contain these intervals in both models. Therefore, $(r,b) \in E_n(\mathcal{I}')$ and $(r,b) \in E_n(\mathcal{I}(B',R'))$. Now, suppose to the contrary that $N(r) \subseteq N(b)$. In both models, b's right endpoint must end to the right of the rightmost left endpoint of neighbors of r. Since both models are red extensions, r's left endpoint lies to the left of the right endpoint of any blue interval with this property, hence to the left of b's right endpoint. By a symmetric argument, r's right endpoint lies to the right of b's left endpoint. Therefore, r's interval intersects b's in both models, so $(b,r) \notin E_n(\mathcal{I}')$ and $(b,r) \notin E_n(\mathcal{I}(B',R'))$.

For the second part of the second claim, suppose $(b,r) \in A_s(\mathcal{I}')$. Then for each neighbor b' of r, $(r,b') \in A_s(\mathcal{I}')$ and $(r,b') \in A_s(\mathcal{I}(B',R'))$, since they are both red extensions. By transitivity of $A_s(\mathcal{I}')$, $(b,b') \in A_s(\mathcal{I}')$. Since both models have the same intersection labels among pairs of blue vertices, $(b,b') \in A_s(\mathcal{I}(B',R'))$. Therefore, the left endpoint of b's interval in $\mathcal{I}(B',R')$ lies to the right of the rightmost left endpoint of a neighbor of r. By the construction of the red extension, r's left endpoint lies to the left of the left endpoint of any such blue interval, hence to the left of the left endpoint of b's interval. Similarly, its right endpoint lies to the right of the right endpoint of b's interval, and $(b,r) \in A_s(\mathcal{I}(B',R'))$. The proof of the converse is identical, except for reversal of the roles of $\mathcal{I}'$ and $\mathcal{I}(B',R')$, since it makes use only of the fact that they are both red extensions and not that $\mathcal{I}(B',R')$ is a co-TT model.

Lemma 13. *Algorithm 2 is correct.*

Proof. Since every red vertex in a co-TT model is simplicial, the preconditions imply that every vertex in R' is simplicial.

By Lemma 10, the labeling of intersection types conducted by the `for` loop on blue pairs is consistent with those in a co-TT model $\mathcal{I}(B',R')$. Therefore, $\mathcal{I}_B$ gives the same intersection labels as $\mathcal{I}_1(B',R')[B']$ does. This is true also for the red extension $\mathcal{I}_2(B',R') = \mathcal{I}(B',R')[B']$, which, by Proposition 2 and Definition 6 is a normalized co-TT model of G'. By Lemmas 11 and 12, those members (x,y) of $A_{V'}$ such that at least one of x and y is a member of R' are assigned to E_n, A_s or $A_{V'}$ in the next two loops if and only if they have those intersection types in $\mathcal{I}_2(B',R')$. Since $\{E_n(\mathcal{I}_2(B',R')), A_s(\mathcal{I}_2(B',R')), A_c(\mathcal{I}_2(B',R')), E_o(\mathcal{I}_2(B',R'))\}$ is a partition of $A_{V'}$, any elements not yet assigned must belong to $E_o(\mathcal{I}_2(B',R'))$, and the final loop correctly assigns these.

The intersection labels assigned to $A_{V'}$ are those of $\mathcal{I}_2(B',R')$. The set $\mathcal{I}_3$ of intervals given by Lemma 8 has these intersection types. Since $\mathcal{I}_2(B',R')$ is a co-TT model, so is $\mathcal{I}_3(B',R')$, and this is the model returned by the algorithm.

Lemma 14. *Algorithm 2 can be implemented to take $O(n^2)$ time even when (G',B',R') does not meet the preconditions.*

Proof. The application of Lemma 7 requires that G' be strongly chordal. This can be checked before the lemma is applied, by Theorems 2 and 4, and G' can be rejected as a co-TT graph if it fails this test, by Theorem 1.

Otherwise, the lemma gives the required neighborhood containments, whether or not G is a co-TT graph, in $O(n^2)$ time. The loop at Line 7 takes $O(n^2)$ time, whether or not G is a co-TT graph. Lemma 8 either gives a set of $\mathcal{I}_B$ of intervals that realizes this labeling, or determines that none exists, in $O(n^2)$ time. By Lemma 10, such a set exists if (G', B', R') meets the precondition, so (G', B', R') can be rejected as failing to meet the preconditions. By Lemma 10, the intersection types of $\mathcal{I}_B$ are the same as they are for some co-TT model $\mathcal{I}(B', R')$ if (G', B', R') meets the preconditions.

Constructing a red extension $\mathcal{I}'$ of $\mathcal{I}_B$ or determining that none exists takes $O(n^2)$ time by elementary methods. By Lemma 9, there is a red extension of $\mathcal{I}_B$ if the preconditions are met, so if there is no red extension, (G', B', R') can be rejected as not meeting the precondition.

Otherwise the time required for the remaining loops do not depend on any additional assumptions about the inputs, and they takes $O(n^2)$ time. The final application of Lemma 8 takes $O(n^2)$ time whether or not it succeeds in producing a set of intervals that realizes the labeling.

Theorem 5. *Recognition of threshold tolerance and co-TT graphs takes $O(n^2)$ time.*

Proof. The problems reduce to each other in $O(n^2)$ time, so we show the result for co-TT graphs. Let G be a graph passed to Algorithm 1. Whether or not G is a co-TT graph, the algorithm halts in $O(n^2)$ time, by Lemmas 4 and 14. If G is a co-TT graph, it returns a co-TT model of G by Lemmas 4 and 13. If G is not a co-TT graph, it produces an incorrect co-TT model, since no co-TT model of G exists, or else it halts without producing one. If it halts without producing one, G can be rejected as a co-TT graph. If the algorithm produces a co-TT model, it takes $O(n^2)$ time to check whether it is a valid co-TT model for G, and if it is, G can be accepted, and if it is not, it can be rejected, since this only happens when G is not a co-TT graph.

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