Nesting of Prime Substructures in k-ary Relations

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Abstract

The **modular decomposition** of a graph or relation has a large number of combinatorial applications. It divides the structure into a set of "prime" induced substructures, which cannot be further decomposed. Recent work on graphs and k-ary relations has focused on the discovery that prime induced substructures are densely nested when they occur. Lower bounds on the "nesting density" of prime substructures in graphs are used heavily in the only known linear-time algorithm for directed graphs. We improve on the previously known lower bounds for k-ary relations, and show that no further improvement is possible.

1 Introduction

Given a finite undirected graph $G = (V, E)$, let $V(G)$ denote V, and let $n(G)$ denote $|V(G)|$, let $G|X$ denote the subgraph of G induced by X, let $G - X$ denote $G|(V(G) - X)$, and $G - x$ denote $G - \{x\}$. Two sets X and Y **overlap** if $X \cap Y$, $X - Y$, and $Y - X$ are all nonempty.

A **module** of G is a set X of nodes such that for any node x not in X, either x is adjacent to every node of X, or x is nonadjacent to every node of X. Figure 4.1 depicts a graph and its modules. The following is easily verified:

Preprint submitted to Elsevier Preprint 23 November 1999

Theorem 1.1 *If X and Y are disjoint modules of a graph, then either every element of X is adjacent to every element of Y, or no element of X is adjacent to any element of Y.*

Thus, any pair of disjoint modules can be classified as "adjacent" or as "nonadjacent." It follows that if $\mathcal{P}$ is a partition of the nodes of G such that each member of $\mathcal{P}$ is a module, the adjacency relationships of the members of $\mathcal{P}$ to each other is itself described by a graph, as shown in Figure 4.2. This graph is called the **quotient** $G/\mathcal{P}$, and $\mathcal{P}$ is called a **congruence partition**. Note that if X is a set obtained by selecting one representative node from each member of $\mathcal{P}$, then $G|X$ is isomorphic to $G/\mathcal{P}$.

The quotient $G/\mathcal{P}$ completely specifies those edges of the graph that are not in any subgraph $G|X$ induced by any $X \in \mathcal{P}$. Thus, the quotient, together with the subgraphs induced by the members of $\mathcal{P}$, gives a complete representation of the original graph.

The **modular decomposition** is a way to represent compactly all modules of a graph. A **strong module** is a module that overlaps no other. The decomposition is a rooted tree. The nodes of this tree are the strong modules of the graph, and the transitive reduction of the containment relation on the strong modules gives the edges of the tree.

An equivalent definition of the modular decomposition is the following recursive one. Note that at least one of G and its complement is connected.

(1) If G is disconnected, its connected components are a congruence partition, every union of components is a module, and no module overlaps a component. Thus, the modules of G can be divided into two sets: those that are a union of components and those that are a subset of a single component. Those that are a subset of a component C can be found by recursion on $G|C$.
(2) If the complement of G is disconnected, apply step 1 to the complement.
(3) Otherwise, let the **highest submodules** be those modules that are not contained in any other module except $V(G)$. When both G and its complement are connected, the highest submodules are a congruence partition. The modules of G are $V(G)$ and those modules that are subsets of highest modules. Those that are a subset of a highest submodule M may be found by recursion on $G|M$.

At each step of the recursion, this algorithm finds a congruence partition. The recursion tree, together with the quotients induced by these congruence partitions, uniquely specify G. Figure 4.3 gives an example.

A variant of the decomposition exists for directed graphs also. The modular decomposition was described in the 1960's by Sabidussi [14] and Gallai [8], and has since been rediscovered independently by different researchers. Kelly [9] gives a history of the idea. Linear-time algorithms have only recently become available [11,4,10]. One of the reasons for interest in the decomposition is the large number of applications to combinatorial problems on graphs and partial orders; Moehring [12] gives a review.

$V(G)$ and its one-element subsets are the *trivial modules*. A prime graph is a graph that has only trivial modules. In dealing with graphs, some people have adopted the convention of considering a structure to be prime only if it has at least three vertices, but we do not place that restriction here. Each quotient in the modular decomposition is either an independent set (step 1), a clique (step 2), or a prime graph (step 3). The quotients produced by step 1 and step 2 are simple; those prime quotients produced by step 3 may be arbitrary prime graphs. This motivates an effort to better understand the structure of prime graphs.

Early in this line of research, it became apparent that any prime undirected graph has a P_4 as an induced subgraph. (A P_4 is a chordless path on four vertices.) A P_4 is the only prime undirected graph with fewer than five vertices. Sumner [16] discovered that this is part of a more general phenomenon, which is that prime undirected graphs contain a long succession of nested prime subgraphs. In particular, he showed that a **hook sequence** exists for every prime undirected graph. A hook sequence for a graph G is a sequence $G_1, G_2, \ldots, G_m$ such that G_1 is a P_4, G_m is G, and each G_i is a prime graph which differs from G_{i-1} by having one more node or the same number of nodes and one more edge. He gives a number of other interesting results about substructures contained within prime undirected graphs.

Definition 1.2 *A* **prime sequence** *for a graph G is a sequence $(G_1, G_2, \ldots G_m)$ of graphs that have only trivial modules, where G_1 has two nodes, and for each $i : 1 \leq i < m$, G_i is an induced proper subgraph of G_{i+1}. A* **maximal prime sequence** *is one where for each $i < m$, there is no prime graph G' such that G_i is an induced proper subgraph of G' and G' is an induced proper subgraph of G_{i+1}. The* **order separation** *of a prime sequence is the maximum of $n(G_{i+1}) - n(G_i)$ over all $i : 1 \leq i < n(G)$, or undefined if $m = 1$.*

Ehrenfeucht and Rozenberg give the following:

Theorem 1.3 *[6]: Every prime graph has a prime sequence whose order separation is at most 2.*

This theorem does not settle the question of whether it is a tight upper bound. Bonizzoni [1] showed that it is tight, but that for any number n of vertices, there are at most four prime graphs that do not have prime sequences with order separation equal to one.

Schmerl and Trotter [15] give the following stronger version of Theorem 1.3.

Theorem 1.4 *In a prime graph, every maximal prime sequence has order separation at most two.*

This shows that *every* prime induced subgraph is contained in a prime sequence of order separation at most two. This theorem is essential to Cournier and Habib's linear-time decomposition algorithm for directed graphs [4]. The difficult step in modular decomposition algorithms is finding maximal induced prime subgraphs, as these are the prime quotients in the modular decomposition. Their algorithm finds them by starting with an arbitrary minimal induced prime substructure, which, by the theorem, must have three or four vertices, and finds a maximal induced prime subgraph that contains it, by inductively finding G_{i+1} from G_i, and using the fact that they differ by at most two vertices.

A k-ary relation R on V is a set of k-tuples on V. A graph is a binary relation. The subrelation induced by $S \subseteq V$ are just those k-tuples in R that are elements of $S \times S \times ... \times S = S^k$. Moehring has given a generalization of modules and the modular decomposition to k-ary relations [13], which we describe in detail in the next section. As in the case of graphs, a module is a subset M of V satisfying certain constraints about the k-tuples containing elements of M and $V - M$, and in the case where $k = 2$, the constraints on the two-tuples (edges) give the definition of a module in a graph.

Given these definitions, a prime sequence, a maximal prime sequence, and its order separation may be defined as in the case of graphs, except that the modules of the k-ary relation take the place of the modules of a graph.

Ehrenfeucht and McConnell give the following generalization to k-ary relations of Theorem 1.3:

Theorem 1.5 *[5] Every prime k-ary relation has a prime sequence whose order separation is at most k.*

Note that, because of the way we have defined a prime relation, the smallest element in a prime sequence may be a relation on just two elements.

1.1 New Results

Ehrenfeucht and McConnell were unable to show that there exist k-ary relations with minimum order separation greater than $k-2$. They left as an open question whether $k-2$, $k-1$, or k is the tightest upper bound. In this paper, we show that it is $k-1$:

Theorem 1.6 *If R is a prime k-ary relation, where $k \geq 3$, then R has a prime sequence of order separation at most $k-1$. This bound is tight; for each $k \geq 3$, there exists a k-ary relation where every prime sequence has order separation* at least $k-1$.

Thus, those k-ary relations that have no prime induced substructures may be characterized by the absence of a (large) blocking set of induced prime substructures on $k+2$ or fewer vertices. Relations with no prime induced substructures give a generalization of cographs and series-parallel partial orders. Two-structures that have no prime induced substructures are studied in [7].

Though Theorem 1.5 is a generalization of Theorem 1.3, it is not a generalization of the much stronger Theorem 1.4. We generalize Theorem 1.4 with the following:

Theorem 1.7 *If R is a prime k-ary relation, where $k \geq 3$, then every maximal prime sequence has order separation at most k. This bound is tight; for each $k \geq 3$, there exist k-ary relations with maximal prime sequences of order separation k.*

Key elements of these new results appeared in Bonizzoni [2].

2 Preliminaries

In this section, we summarize definitions and previous results that we use in the remainder of the paper.

If X and Y are two sets, the **symmetric set difference** $X \Delta Y$ denotes those elements that are in exactly one of X and Y. Let $X-y$ denote $X-\{y\}$ and $X+y$ denote $X+\{y\}$.

Definition 2.1 *A k-ary relation R on V consists of a subset of V^k, that is, a subset of the k-tuples on V. V may be denoted $V(R)$. For $e \in R$, $V(e)$ denotes the set of elements of $V(R)$ that occur in e. Note that an element of $V(R)$ may occur* *multiple times in e. Two elements of $(V(R))^k$ are the* **same color** *under R if they are both members of R or both members of $(V(R))^k - R$. A k-ary relation may thus be viewed as a coloring of V^k with two colors. A* **k-structure** *is a coloring of V^k with an arbitrary number of colors, and thus a generalization of k-ary relations. If $X \subseteq V(R)$, then the* **subrelation**, *or* **substructure**, *of R induced by X, denoted $R|X$, is a relation R' such that $V(R') = X$, and $R' = R \cap X^k$. Let $R - X$ denote $R|(V(R) - X)$, and $R - x$ denote $R|(V(R) - x)$. Two k-ary relations R and R' are* **isomorphic** *if there is a bijection ϕ from $V(R)$ to $V(R')$ such that $(v_1, v_2, ..., v_k) \in R$ if and only if $(\phi(v_1), \phi(v_2), ..., \phi(v_k)) \in R'$. The* **reflexive** *elements of V^k are those of the form $(v, v, v, ..., v)$. An* **antireflexive** *k-ary relation is one where R does not contain any reflexive elements.*

Antireflexive k-ary relations give a generalization of loopless graphs and digraphs, which are antireflexive 2-ary (binary) relations. For this reason, we will call the members of $V(R)$ the **vertices** and the members of R the **edges** of R. The following gives a generalization of their modules.

Definition 2.2 *If $X \subseteq V(R)$ we define a relation R_X on $(V(R))^k$: let $(v_1, v_2, ..., v_{i-1}, y, v_{i+1} ..., v_k)$ R_X $(v_1, v_2, ..., v_{i-1}, z, v_{i+1}, ..., v_k)$ if y and z are both elements of X and at least one element of these k-tuples is an element of $V(R) - X$. The X* **classes** *are the equivalence classes on $(V(R))^k$ induced by the transitive reflexive closure of R_X. A* **module** *in a k-ary relation is a set X such that each X class on the edges is the same color under R. A module of a k-structure has an identical definition.*

Example 2.3 *Let R be a 4-ary relation on domain $\{1, 2, 3, 4, 5\}$. Suppose $\{1, 2, 3\}$ is a module. Then $(2, 4, 2, 1)$ is an edge if and only if $(2, 4, 3, 1)$ is, since these are two edges that contain elements of X and $V(R) - X$, and differ by replacement of one element of X with another. It is possible that $(2, 4, 2, 1)$ is an edge and $(2, 5, 2, 1)$ is not, since these differ by an element not contained in X.*

Theorem 2.4 *If R is a k-ary relation, then $V(R)$ and the members of $\{\{v\} : v \in V(R)\}$ are modules of R.*

Definition 2.5 *The modules given by Theorem 2.4 are the* **trivial modules**. *The empty set is sometimes considered to be a (trivial) module. A k-ary relation is* **prime** *if it has at least three vertices and has only trivial modules.*

Restriction of the definition to symmetric binary relations gives exactly the definition of a module on an undirected graph. Restriction to arbitrary binary relations gives the usual definition of a module in a directed graph [12]. A congruence partition is a partition $\mathcal{P}$ of $V(R)$ such that every member of $\mathcal{P}$ is a module. The congruence partitions whose members are all trivial modules are the **trivial congruence partitions**. The quotient is defined as in the case of graphs, that is, it is a k-ary relation on $\mathcal{P}$ that is isomorphic to $R|S$ for an arbitrary set S consisting of one representative from each member of $\mathcal{P}$. The exact selection of S makes no difference; if S_1 and S_2 are two such selections, $R|S_1$ and $R|S_2$ are isomorphic, since $\mathcal{P}$ is a congruence partition.

Definition 2.6 *The* **complement** *of a k-ary antireflexive relation R is a relation $\overline{R}$, with $V(\overline{R}) = V(R)$, and $\overline{R}$ consisting of the nonreflexive elements of $(V(R))^k - R$. Two vertices are* **adjacent** *if they are contained in some edge. Two vertices are in the same* **connected component** *if they are joined by a* **path**, *which is a sequence $(x = v_1, v_2, ..., v_j = y)$ of vertices where consecutive vertices are adjacent.*

Moehring [13] shows that the modular decomposition applies under these definitions. He also gives the following theorems for k-ary relations:

Theorem 2.7 *(The "Overlap Rule") If X and Y are overlapping modules of a k-ary relation R, where $k \geq 3$ then $X \cap Y$, $X \cup Y$, $X - Y$, $Y - X$, and $Y \Delta X$ are all modules of R.*

Technically, the version of Theorem 2.7 given in [13] does not contain the restriction that $k \geq 3$, and omits that $Y \Delta X$ is a module, because this is not true when $k = 2$. The reader may find a proof of this straightforward observation for $k \geq 3$ in [5].

Theorem 2.8 *(The "Restriction Rule") Let R be a k-ary relation, let X be a module of R, and let Y be a subset of $V(R)$. Then $X \cap Y$ is a module of $R|Y$.*

Theorem 2.9 *(The "Autonomous Substructure Rule") Let R be a k-ary relation and let Y be a module of R. Then $X \subseteq Y$ is a module of $R|Y$ if and only if it is a module of R.*

Theorem 2.10 *(The "Quotient Rule") Let R be a k-ary relation, and let $\mathcal{P}$ be congruence partition on G. Then $\mathcal{Y}$ is a module of $R/\mathcal{P}$ if and only if $\bigcup \mathcal{Y} \subseteq \mathcal{P}$ is a module of R.*

It should be noted that Ehrenfeucht and McConnell proved that Theorem 1.5 is valid for arbitrary k-structures, not just k-ary relations. They prove it directly on k-structures. We get a similar generalization of Theorems 1.6 and 1.7 by a reduction argument, which allows us to ignore the additional properties of k-structures in the remainder of the paper:

Theorem 2.11 *Theorems 1.6 and 1.7 are valid for k-structures.*

Proof: We prove their correctness for k-ary relations below. Let g be a k-structure, namely a coloring of the nonreflexive elements of V^k. Without loss of generality, assume the colors are $\{0, 1, ..., C-1\}$. For each i from 1 to $\log_2 C$, we define a k-ary relation R_i. R_i contains a nonreflexive element e of V^k as an edge if and only if the binary representation of the color of e in g has a 1 in the i^{th} position. Clearly, a set $X \subseteq V$ is a module in g if and only if it is a module in each R_i. Thus, if $R_i|P$ is prime for some $P \subseteq V$, then $g|P$ is also prime. An upper bound on the order separation of prime sequences in any R_i is thus an upper bound on the order separation of prime sequences in g.

A k-ary relation is a special case of a k-structure, so if an upper bound on order separations in k-ary relations is tight, the same bound must be tight in k-structures.

3 Proofs of Theorems 1.6 and 1.7

Definition 3.1 *Let R be a k-ary relation on V. The set of* **maximal prime sets** *of R, $\mathcal{M}(R)$, is the family of sets $\{X : X \subset V,\ R|X$ is prime, and for all Y such that $X \subset Y \subset V$, $R|Y$ is not prime$\}$. The* **maximum prime sets** *are those sets of $\mathcal{M}(R)$ that are as large as any other.*

Because our definition of prime relations admits relations on two elements, a members of $\mathcal{M}(R)$ may have two elements.

Theorem 1.7 follows from this claim:

Claim 1 *Let R be a prime k-ary relation, where $k \geq 3$.*

(1) Every member of $\mathcal{M}(R)$ has cardinality at least $|V(R)| - k$.
(2) If $|V(R)| > k + 1$, then $\mathcal{M}(R)$ is nonempty.
(3) For each $k \geq 3$, there exists a prime k-ary relation S where $\mathcal{M}(S)$ has members of size at most $|V(S)| - k$.

The first relation of a prime sequence is an arbitrary induced two-vertex k-ary relation. Statement 2 shows the existence of the second. The remainder of the sequence follows by induction, using Statement 1. Statement 3 shows that no tighter bound is possible.

The proof of Theorem 1.6 then follows from the next claim:

Claim 2 *Let R be a prime k-ary relation, where $k \geq 3$.*

(1) Some member of $\mathcal{M}(R)$ has cardinality at least $|V(R)| - (k-1)$.
(2) There exists a prime k-ary relation S where no member of $\mathcal{M}(S)$ has cardinality greater than $|V(S)| - (k-1)$.

The bound of Theorem 1.6 follows by induction from Statement 1 of Claim 2, together with Statement 2 of Claim 1, which serves as a base case. The tightness follows from Statement 2.

We prove the five statements of Claims 1 and 2 one-by-one in the following subsections. By the foregoing arguments, this suffices to prove Theorems 1.6 and 1.7.

3.1 Preparations for proving Statement 1 of Claims 1 and 2

In this section, we give some basic definitions and results that are needed for the proofs of Statement 1 of Claims 1 and 2. In particular, we show that the ways in which nontrivial modules can appear when new vertices are added to prime structures is quite constrained.

First, we give a generalization of some concepts that are due to [6], which develops them in the context of binary relations and two-structures.

Definition 3.2 *Let R be a k-ary relation, and let X be a subset of $V(R)$ such that $R|X$ is prime. For $x \in X$ and $y \in V(R) - X$, x and y are* **locally equivalent** *if $\{x, y\}$ is a module in $R|(X + y)$. If X is a module in $R|(X + y)$, then y is* **global to X**. *If $R|(X + y)$ is prime, then y* **extends** *X. Let S_X be the relation on $V(R)$ where aS_Xb if a and b are locally equivalent to each other or to a common vertex x, or if they are both global to X, or if they both extend X. The X* **classes** *are the equivalence classes given by the transitive reflexive closure of S_X.*

Lemma 3.3 *[5] If R is a k-ary relation, $X \subseteq V(R)$, and $R|X$ is prime, then the members of X are in disjoint X classes, the elements that are global to X make up one X class, and the elements that extend X make up one X class.*

Lemma 3.3 justifies the classification of X classes as **local classes**, the **global class**, and the **extension class**. (See Figure 4.4.) The local classes are called "sprouts" in [5]; the terminology we use is suggested by [4].

Definition 3.4 *Let R be a k-ary relation, where $X, Y \subseteq V(R)$. Then Y* **enables** *X if and only if for every element $y \in Y$, $X - y$ is a module of $R - y$.*

Note that the definition of an enabling set does not require that y be an element of X or that Y not be a module of R.

Lemma 3.5 *Suppose Y enables X, and e_1 and e_2 are two edges such that $e_1 R_X e_2$ and $Y - (V(e_1) \cup V(e_2)) \neq \emptyset$. Then e_1 and e_2 are the same color.*

Proof: Let y be an element of $Y - (V(e_1) \cup V(e_2))$; $e_1, e_2 \in (V(R) - y)^k$, and since $X - y$ is a module in $R - y$, e_1 and e_2 are the same color.

Lemma 3.6 *[5] If Y enables X and $|Y| \geq k + 2$, then X is a module of R.*

Proof: For an arbitrary pair $\{e_1, e_2\}$ of elements of $(V(R))^k$ such that $e_1 R_X e_2$, $|V(e_1) \cup V(e_2)| \leq k+1$. Thus, $Y - (V(e_1) \cup V(e_2))$ is nonempty, so the lemma follows from Lemma 3.5.

The following two lemmas are variations of Lemma 3.6 that do not appear in [5], and that are key elements in obtaining the new bounds.

Lemma 3.7 *If Y enables X, $|Y| \geq k + 1$, and $|X| \geq 3$, then X is a module of R.*

Proof: We show that for any pair $e_1, e_2 \in (V(R))^k$ such that $e_1 R_X e_2$, e_1 and e_2 are the same color under R, which implies that X is a module. If $Y - (V(e_1) \cup V(e_2))$ is nonempty, this follows from Lemma 3.5. So assume that $Y - (V(e_1) \cup V(e_2))$ is empty. Since $e_1 R_X e_2$, e_1 and e_2 differ by only one element, so $|V(e_1) \cup V(e_2)| \leq k + 1$, and since $|Y| \geq k+1$, $Y = V(e_1) \cup V(e_2)$, all elements of e_1 are distinct, and all elements of e_2 are distinct. Let j be the position where e_1 and e_2 differ, let x_1 be the j^{th} element of e_1 and let x_2 be the j^{th} element of e_2. Let e_3 be the element of V^k obtained by replacing the j^{th} element of one of these vectors with $x_3 \in X - \{x_1, x_2\}$. This ensures that $e_1 R_X e_3$ and $e_2 R_X e_3$. Whether or not $x_3 \in Y$, $Y \cap (V(e_1) \cup V(e_3)) = k$, and $Y \cap (V(e_2) \cup V(e_3)) = k$. $Y - (V(e_1) \cup V(e_3))$ and $Y - (V(e_2) \cup V(e_3))$ are nonempty. By Lemma 3.5, e_1 and e_3 are the same color, as are e_2 and e_3. Thus, e_1 and e_2 are the same color.

Lemma 3.8 *If R is a k-ary relation, $|Y| \geq k$, Y enables X, and Y is either a subset of X or is disjoint from X, then X is a module in R.*

Proof: Again, it suffices to show that if $e_1, e_2 \in (V(R))^k$ and $e_1 R_X e_2$, then e_1 and e_2 are the same color. Suppose Y is disjoint from X. Since $|V(e_1) - X| \leq k - 1$ and $V(e_1) - X = V(e_2) - X$, $Y - (V(e_1) \cup V(e_2))$ is nonempty. By Lemma 3.5, e_1 and e_2 must be the same color. So suppose that $Y \subseteq X$. If $Y - (V(e_1) \cup V(e_2))$ is nonempty, they are again the same color by Lemma 3.5. So assume that $Y - (V(e_1) \cup V(e_2))$ is empty. Let j be the position where e_1 and e_2 differ, let x_1 be the j^{th} element of e_1 and let x_2 be the j^{th} element of e_2. Since e_1 and e_2 both have an element in

$V(R)-X$, $|(V(e_1)\cup V(e_2))\cap X| \leq k$, and since $|Y| \geq k$, $Y = (V(e_1)\cup V(e_2))\cap X$, all elements of e_1 are distinct, and all elements of e_2 are distinct. Let e_3 be the k-tuple obtained by substituting $x_3 \in X - \{x_1, x_2\}$ for x_1 in e_1. Whether or not $x_3 \in Y$, $Y \cap (V(e_1) \cup V(e_3)) = k-1$, and $Y \cap (V(e_2) \cup V(e_3)) = k-1$. $Y - (V(e_1) \cup V(e_3))$ and $Y - (V(e_2) \cup V(e_3))$ are nonempty. By Lemma 3.5, e_1 and e_3 are the same color, as are e_2 and e_3. Thus, e_1 and e_2 are the same color.

Lemma 3.9 *[5] Let R be a non-prime k-ary relation, where $k \geq 3$, and let $M \in \mathcal{M}(R)$. Each local M class on $V(R)$ (see Definition 3.2) is a module in R, and the union of the local M classes is a module in R.*

Lemma 3.10 *[5] Let R be a k-ary relation, where $k \geq 3$, and let $M \in \mathcal{M}(R)$. Then $V(R) - M$ enables each local M class, as well as the union of the local equivalence classes of M.*

Lemma 3.11 *Let R be an arbitrary k-ary relation, where $k \geq 3$. Suppose M is a maximum-cardinality member of $\mathcal{M}(R)$, and $|M| \leq |V(R)| - 3$. Let A be a non-singleton local M class, let m be the member of $A \cap M$, let $x \in A - M$, and let $M' = (M - m) + x$. Then:*

(1) $R|M'$ is isomorphic to $R|M$;
(2) $M' \in \mathcal{M}(R)$;
(3) The local M' classes are the same as the local M classes.

Proof: By Lemma 3.3, m is unique. Since $\{m, x\}$ is a module in $R|(M+x)$, $R|M'$ is isomorphic to $R|M$ by the Quotient Rule (2.10). Thus, $R|M'$ is also prime, and since it is just as large as $R|M$, it has cardinality as great as that of any member of $\mathcal{M}(R)$, so $M' \in \mathcal{M}(R)$.

It remains to show that M' has the same local equivalence classes as M does. Suppose there exists another member y of the local equivalence class containing m. Then $\{m, x, y\}$ is a local M class in $R|(M \cup \{x, y\})$. Since $|M| < |M \cup \{x, y\}| < |V(R)|$, $R|(M \cup \{x, y\})$ is non-prime, hence $\{m, x, y\}$ is a module in $R|(M \cup \{x, y\})$, by Lemma 3.9. By the Restriction Rule (2.8), $\{x, m\}$ is a module in $R|(M' + m)$ and $\{x, y\}$ is a module in $R|(M' + y)$. Thus, m and y are in x's local M' class, and y remains in the same local equivalence class as m. Since y was an arbitrary member of m's M class, this class must be a subset of m's M' class. By symmetry, m's M' class is a subset of m's M class. The two classes must be equal.

Similarly, suppose there is a local M class Z not containing m, and let $z_1, z_2 \in Z$ with $z_1 \in M$. Since $|M \cup \{x, z_2\}| < |V(R)|$ and $M \in \mathcal{M}(R)$, $R|(M \cup \{x, z_2\})$ is not prime and $M \in \mathcal{M}(R|(M \cup \{x, z_2\}))$. By Lemma 3.9, $\{z_1, z_2\}$ is a module in

$R|(M \cup \{x, z_2\})$. However, $M \cup \{x, z_2\} = M' \cup \{m, z_2\}$, so $\{z_1, z_2\}$ is a module in $R|(M' \cup \{m, z_2\})$, hence a module in $R|(M' + z_2)$, by the Restriction Rule (2.8). Thus, z_1 and z_2 are in the same M' class. Since z_2 was an arbitrary member of z_1's M class, this class is a subset of z_1's M' class. By symmetry, its M' class is a subset of its M class, as required.

The following lemma enlarges the enabling set of Lemma 3.10 so that it includes the members of M that are members of non-singleton local M classes. To do this, we must assume M is not only a member of $\mathcal{M}(R)$, but a maximum-cardinality member of $\mathcal{M}(R)$. The necessity of the restriction will explain why the bound of $k-1$, given in Theorem 1.6, does not hold in the more general setting of Theorem 1.7.

Lemma 3.12 *Suppose R is a k-ary relation, M is a maximum-cardinality member of $\mathcal{M}(R)$ and $|V(R)| - |M| \geq 3$. Let X be the union of the global M class and the non-singleton local M classes. Then X enables each local M class and the union of the local M classes.*

Proof: Lemma 3.10 establishes that $V(R) - M$ enables each local equivalence class and the union of local equivalence classes. Each element of $V(R) - M$ is either in the global class or in a non-singleton local class. It remains to show that the members of M that are in non-singleton local classes also enable these sets. Let m be an arbitrary member M that is in a non-singleton local equivalence class, let x be another member of the class, and let $M' = (M - m) + x$. By Lemma 3.10 applied to M', m enables the local equivalence classes of M', as well as their union. By Lemma 3.11, the local equivalence classes of M' are the same as those of M. Thus, m enables the required sets.

3.2 Proof of Statement 1 of Claim 1

Lemma 3.13 *Let R be a prime k-ary relation, where $k \geq 3$. Then every member of $\mathcal{M}(R)$ has cardinality at least $|V(R)| - k$.*

Proof: Suppose for the purpose of contradiction that R is prime and for some $M \in \mathcal{M}(R)$, $|V(R) - M| \geq k + 1$.

If M has an element x in the extension class, $R|(M + x)$ is prime, which contradicts M's membership in $\mathcal{M}(R)$.

Otherwise, if the global M class is empty and there is a local class A of size at least three, then by Lemma 3.7 and 3.10, A is a nontrivial module, a contradiction. Otherwise, all local equivalence classes are of size two. Let B be such a local equivalence class. By Lemma 3.10, the set $Y = V(R) - M - B$ enables B and is disjoint from it. By Lemma 3.8, B is a module of R, a contradiction.

Otherwise, if the global M class is nonempty, then $V(R) - M$ is a set of size $k+1$ that enables the union U of the local equivalence classes, by Lemma 3.10. Since $|M| \geq 3$, this U has at least three elements, hence it is a module of R, by Lemma 3.7. Since the global equivalence class is nonempty, $U \neq V(R)$, and it is a nontrivial module of R, a contradiction.

By Lemma 3.3, this exhausts all the cases.

3.3 Proof of Statement 1 of Claim 2

Lemma 3.14 *Let R be a prime k-ary relation, where $k \geq 3$. All maximum-cardinality members of $\mathcal{M}(R)$ have cardinality at least $|V(R)| - (k-1)$.*

Proof: Let M be a *maximum-cardinality* member of $\mathcal{M}(R)$. For the purpose of contradiction, assume $|V(R) - M| \geq k$. Since $k \geq 3$, $|V(R) - M| \geq 3$. Let U be the set of elements of M that are in non-singleton local M classes. By Lemma 3.12, the set $Y = (V(R) - M) \cup U$ enables each local M class and the union of local M classes. If $|U| = 0$, the union of the local equivalence classes is M, $|Y| \geq k$ and disjoint from M, and M is a nontrivial module in R, by Lemma 3.8. This contradicts the primality of R. If $|U| \geq 2$, $|Y| \geq k+2$, and there is a non-singleton local M class, which must be a nontrivial module in R by Lemma 3.6. This also contradicts the primality of R. If $|U| = 1$, then $|Y| = k+1$. Let X be the union of local M classes. If $X \neq V(R)$, then $3 \leq |M| \leq |X| < V(R)$, so X is a nontrivial module of R, by Lemma 3.7. Otherwise, the sole local equivalence class, A, must contain U and $V(R) - M$. In this case, $|A| = k+1 > 3$, so A is a nontrivial module of R, by Lemma 3.7.

3.4 Proof of Statement 2 of Claim 1

A k-ary relation R is *degenerate* if it either contains every nonreflexive element of $(V(R))^k$ or it contains none of them. It is easily verified that every subset of $V(R)$ is a module if and only if R is degenerate [5].

Lemma 3.15 *[5] Let R be a non-prime k-ary relation such that $\mathcal{M}(R)$ is empty, and let $X \subseteq V(R)$. Then there are two disjoint modules Y_1 and Y_2 of R such that*

$Y_1 \cap X$ and $Y_2 \cap X$ are a nontrivial partition of X.

Corollary 3.16 *Let R be as in Lemma 3.15. If X is a module, then there are two modules that are a partition of X.*

Proof: $X \cap Y_1$ and $X \cap Y_2$ are modules of R, by the Overlap rule; these two modules are the required partition.

Lemma 3.17 *[5] Let R be a non-prime k-ary relation such that $\mathcal{M}(R)$ is empty. Then R has a doubleton module.*

Lemma 3.18 *[5] Let R be a non-prime k-ary relation such that $\mathcal{M}(R)$ is empty, and let D be a doubleton subset of $V(R)$. Then D is a module in R if and only if it is a module in $R|(D+x)$ for each $x \in V(R) - D$.*

Lemma 3.19 *Suppose R is a k-ary relation on at least $k+2$ vertices and $\mathcal{M}(R)$ is empty. Let $x, y \in V(R)$. If there is a set X that is a nontrivial module in both $R-x$ and $R-y$, then R is non-prime.*

Proof: Since $R|X$ is non-prime, it has a doubleton module D by Lemma 3.17. Since X is a module in both $R-x$ and $R-y$, D is a module in both $R-x$ and $R-y$ by the Autonomous Substructure Rule (2.9). Since D is a module of $R-x$, it is a module in $R|(D+w)$ for each element w of $V(R)-D-x$. Since it is a module in $R-y$, D is a module in $R|(D+x)$. Thus, D is a module in $R-w$, for any $w \in V(R)-D$, by Lemma 3.18. D is enabled by $V(R)-D$. It is also enabled by each $u \in D$, since $D-u$ is a trivial module. Thus, D is enabled by $V(R)$. D must be a module in R by Lemma 3.6.

Lemma 3.20 *Let R be a non-prime k-ary relation such that $\mathcal{M}(R)$ is empty. If $x \in V(R)$, and A is a module of $R-x$, then either A or $A+x$ is a module in R.*

Proof: Suppose neither A nor $A+x$ is a module of R. Let M be a minimal-cardinality module of R that contains A and x. M exists since $V(R)$ is a module that contains A and x. If A or $A+x$ is a module in $R|M$, it is a module in R by the Autonomous Substructure Rule, a contradiction.

Suppose there is no nontrivial module containing x in $R|M$. There is a partition of M into two modules of $R|M$, by the corollary to Lemma 3.15. One of these is $\{x\}$, so the other contains A. By the Autonomous Substructure Rule, A is a module of $R|M$, a contradiction.

Suppose there is a nontrivial module Y of $R|M$ that contains x and is disjoint from A. By the Restriction Rule, applied to $R-x$, A is a module of $R|((M-Y)+y)$

for $y \in Y - x$. However, this is a quotient on $R|M$ obtained by replacing Y with a single representative. By the Quotient Rule, A is a module in $R|M$, a contradiction.

Thus, there is a nontrivial module Z of $R|M$ that contains x and some, but not all, elements of A. By the restriction rule, $A - Z + z$ is a module of $(R|M) - Z + z$, for $z \in Z \cap A$. But $(R|M) - Z + z$ is the quotient on $R|M$ obtained by replacing Z with z. By the quotient rule, $A \cup Z$ is a module of $R|M$ containing A. By the Autonomous Substructure Rule, A is a module of $R|M$, a contradiction.

We may now prove Claim 1, Statement 2.

Lemma 3.21 *Let R be a prime k-ary relation, where $k \geq 3$. If $|V(R)| > k+1$ then $\mathcal{M}(R)$ is nonempty.*

Proof: Suppose that $|V(R)| > k+1$, but that $\mathcal{M}(R)$ is empty. By Lemma 3.17, for any $x \in V(R)$, $R - x$ has a doubleton module D. Let y and z be two elements of $V(R) - (D + x)$. By the Restriction Rule, D is also a module in $R - \{x, y\}$ and $R - \{x, z\}$. By Lemma 3.20, either D or $D + x$ is a module in each of $R - y$ and $R - z$. Either two of $\{R - x,\ R - y,\ R - z\}$ have D as a module, or two of them have $D + x$ as a module. By Lemma 3.19 R is not prime, a contradiction.

3.5 Proof of Statement 3 of Claim 1

Lemma 3.22 *Let R be a k-ary relation, let e be an edge of R, and $X \subseteq V(R)$.*

(1) If $V(e)$ is contained in X or disjoint from X, then X is a module in R if and only if it is a module in $R - e$.
(2) If $|X| \geq 2$, and $V(e)$ intersects X and $V(R) - X$, then X is a module in at most one of R and $R - e$.

Proof: Statement 1 is immediate from the definition of a module. For Statement 2, replace an occurrence of a member of X in e with another member of X. The resulting member $e' \in (V(R))^k$ is in the same X class as e. X can be a module in at most one of R and $R - e$, since e and e' can be the same color in at most one of these relations.

Lemma 3.23 *For each $k \geq 3$, there exists a prime k-ary relation S where $\mathcal{M}(S)$ has members of size at most $|V(S)| - k$.*

Proof: Given a k-ary relation R, let a **doubling** of a vertex $x \in V(R)$ be the result of adding a new vertex x' to $V(R)$, and a minimal set E' of new edges containing x'

to make $\{x, x'\}$ a module in $R' = R \cup E'$. That is, for each $e = (a_1, a_2, ..., a_k) \in R$ such that some $a_i \in V(R) - x$, let $J = \{j : a_j = x\}$. For each subset J' of J, add a new edge $e' = (b_1, b_2, .., b_k)$ such that for each $j : 1 \leq j \leq k$, $b_j = x'$ if $j \in J'$ and $b_j = a_j$ otherwise.

Given an arbitrary prime k-ary relation R on at least k vertices, select a subset $X = \{x_1, x_2, ..., x_k\}$ of vertices of R, and sequentially double each vertex $x_i \in X$, thereby adding a new vertex x'_i, as well as those new edges required to make $\{x_i, x'_i\}$ a module. Let R_2 be the new k-ary relation, and let X' be the k new vertices $\{x'_1, x'_2, ..., x'_k\}$. For each i from 1 to k, $\{x_i, x'_i\}$ is a module in R_2. R is a quotient $R_2/\mathcal{P}$, where the nontrivial members of $\mathcal{P}$ are the sets $\{x_i, x'_i\}$. R is prime, so by the Quotient Rule (2.10), the sets $\{x_i, x'_i\}$ are the only nontrivial modules in R_2. Let S be the result of selecting an element $e \in (V(R_2))^k$ that contains all elements of X', and removing e from R_2 if it is already an edge in R_2, or adding e to R_2 otherwise. By Statement 2 of Lemma 3.22, each $\{x_i, x'_i\}$ is a non-module in S, and S is therefore prime. Let P be a member of $\mathcal{M}(S)$ such that $V(R) \subseteq V(P)$. Without loss of generality, suppose $x'_1 \notin P$. For each $i : 2 \leq i \leq k$, $\{x_i, x'_i\}$ is a module in $S - x'_1$; this follows from the Restriction Rule (2.8) and the fact that $S - x'_1$ is also an induced substructure of R_2. By the Restriction Rule (2.8), $P \cap \{x_i, x'_i\}$ is, therefore, a module in $R|P$. P must contain at most one of $\{x_i, x'_i\}$, hence $|P| \leq |V(S)| - k$.

3.6 Proof of Statement 2 of Claim 2

Lemma 3.24 *There exists a prime k-ary relation where no member of $\mathcal{M}(R)$ has cardinality greater than $|V(R)| - (k - 1)$.*

Proof: Let $k \geq 3$. Define a set $X_i = \{x_{i,1}, x_{i,2}, ..., x_{i,k}\}$ for each i from 1 to arbitrary integer $p \geq 2$. Let R be a relation with $V(R) = \bigcup X_i$. A nonreflexive element of $(V(R))^k$ is an edge in R if and only if it contains some $x_{i,j}$ with $j > k/2$. Note that $A \subseteq V(R)$ is a module in R if and only if it contains no element $x_{i,b}$ with $b > k/2$, or else every element $x_{i,a}$ with $a \leq k/2$, or else no element $x_{i,a}$ with $a \leq k/2$. Next, create a new relation R' from R by removing, for each i, one edge e such that $V(e) = X_i$. R' is now prime, by Statement 2 of Lemma 3.22. Let P be a member of $\mathcal{M}(R')$, and select an arbitrary $x_{i,j} \in V(R') - P$. Let A be the set of elements of $X_i - x_{i,j}$ that have indices between 1 and $k/2$, and let B be the elements of X_i that have indices higher than $k/2$.

If P contains no element of $V(R') - X_i$, then, since $|V(R') - X_i| \geq k$, $|V(R') - P| \geq k$, as required. So suppose that P contains an element of $V(R') - X_i$. Let R'' be the result of adding back to R' the removed edge e of R where $V(e) = X_i$. By Statement

1 of Lemma 3.22, A and $V(R)-B$ are modules in R''. By the Restriction Rule (2.8), $A - x_{i,j}$ and $V(R) - B - x_{i,j}$ are modules in $R'' - x_{i,j} = R' - x_{i,j}$. Again by the Restriction Rule (2.8), $A \cap P$ and $(V(R') - B) \cap P$ are modules in $R'|P$. Since $R'|P$ is prime and contains elements of $V(R') - B$, P must have no element of B and at most one element of A. Since $|A \cup B| = k$, $|P| \leq |V(R')| - (k-1)$, as required.

4 Figures

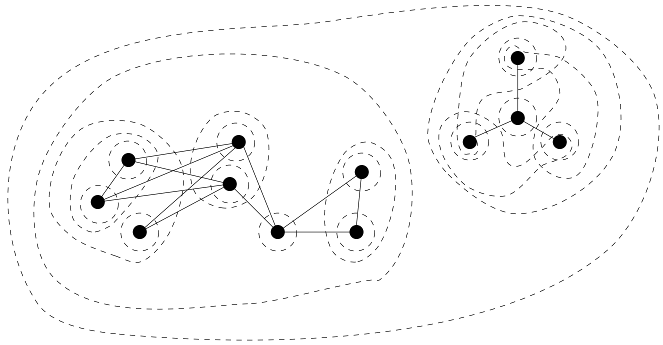

Figure 4.1 *A graph and its modules. A module is a set X of vertices such that for each $y \in V(G) - X$, either y is adjacent to every element of X, or y is nonadjacent to every element of X.*

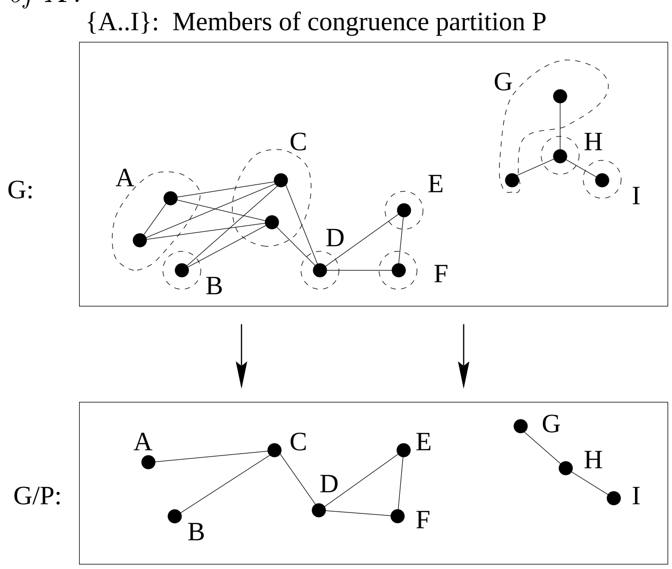

Figure 4.2 *A quotient on the graph of Figure 4.1. For any pair $\{X, Y\}$ of disjoint modules, either every element of $X \times Y$ is an edge or none is. Thus, X and Y may be viewed as* adjacent *or* nonadjacent*. If $\mathcal{P}$ is a partition of the vertices a graph*

into modules, the adjacencies of members of $\mathcal{P}$ can be described by a quotient graph *$G/\mathcal{P}$. The quotient graph and the subgraphs induced by members of $\mathcal{P}$ completely specify G.*

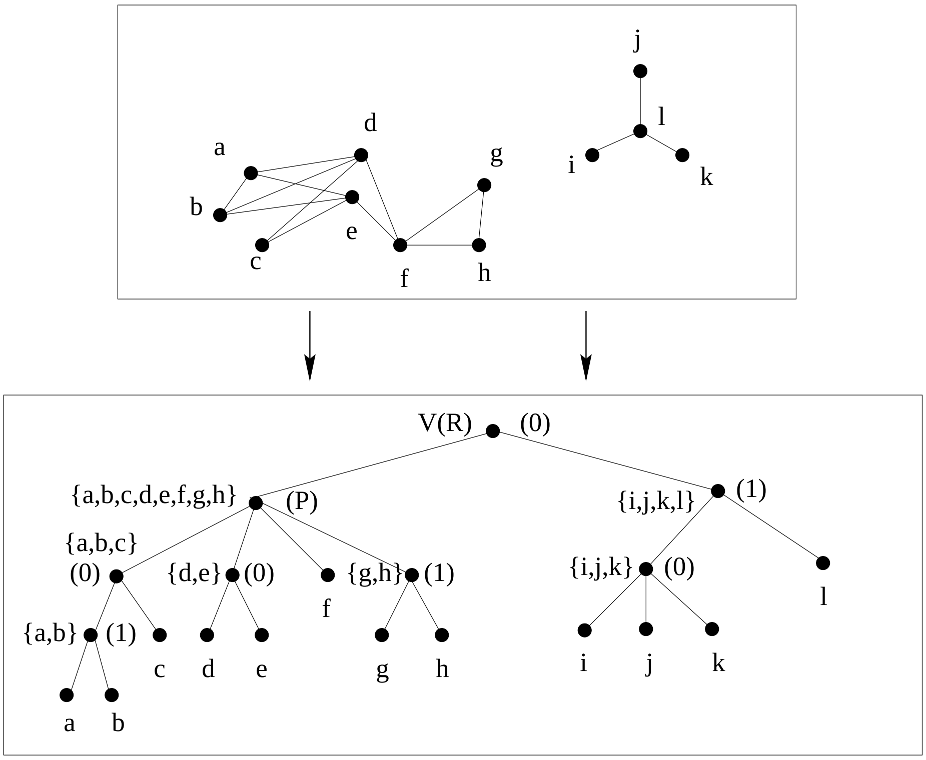

Figure 4.3 *The modular decomposition of the graph of Figure 4.1. The leaves are the vertices of the graph. The internal nodes are the strong modules, and their members are their leaf descendants. An internal node is a* parallel node *(labeled 0) if the set it denotes induces a disconnected graph, and its children are its connected components. It is a* series node *(labeled 1) if the complement of the graph it induces is a disconnected graph, and its children are the connected components of the complement. It is a* prime node *if it induces a graph that is connected and whose complement is connected; its children are the maximal modules it contains. A set of vertices is a module if and only if it is a node of the tree or a union of children of a single parallel or degenerate node.*

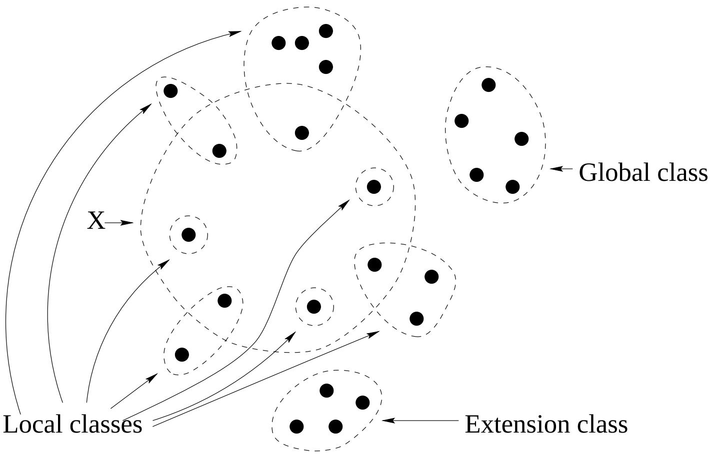

Figure 4.4 *Suppose $X \subseteq V(R)$ and $R|X$ is prime. The* global class *is* $\{y : y \in V(R)$ *and* X *is a module in* $R|(X+y)\}$. *The* extension class *is* $\{y : y \in V(R)$ *and* $R|(X+y)$ *is prime* $\}$. *For each* $x \in X$, *the* local class *containing* x *is* $\{y : \{x\} \cup \{y\}$ *is a module in* $R|(X+y)\}$. *(This class always contains* x.*) These classes partition* $V(R)$.

5 Conclusions

In this paper, we show that every prime k-ary relation, where $k > 3$, has a prime sequence of order separation at most $k-1$. This is not true for binary relations [6,15,1].

We also show that that every prime k-ary relation is contained in a prime sequence of order separation at most k, and that this bound is tight for every $k > 2$. This implies that for every prime induced subrelation R', a prime sequence starting at R' may be constructed by adding at most k vertices at a time to obtain another prime induced k-ary relation. We show a reduction by which this same result extends to arbitrary k-structures. An analogous observation is the basis of the modular decomposition algorithm of [3] for graphs, and we think that a similar technique might be applied to modular decomposition of k-ary relations.

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